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Alfred Gray

Tubes

Second Edition

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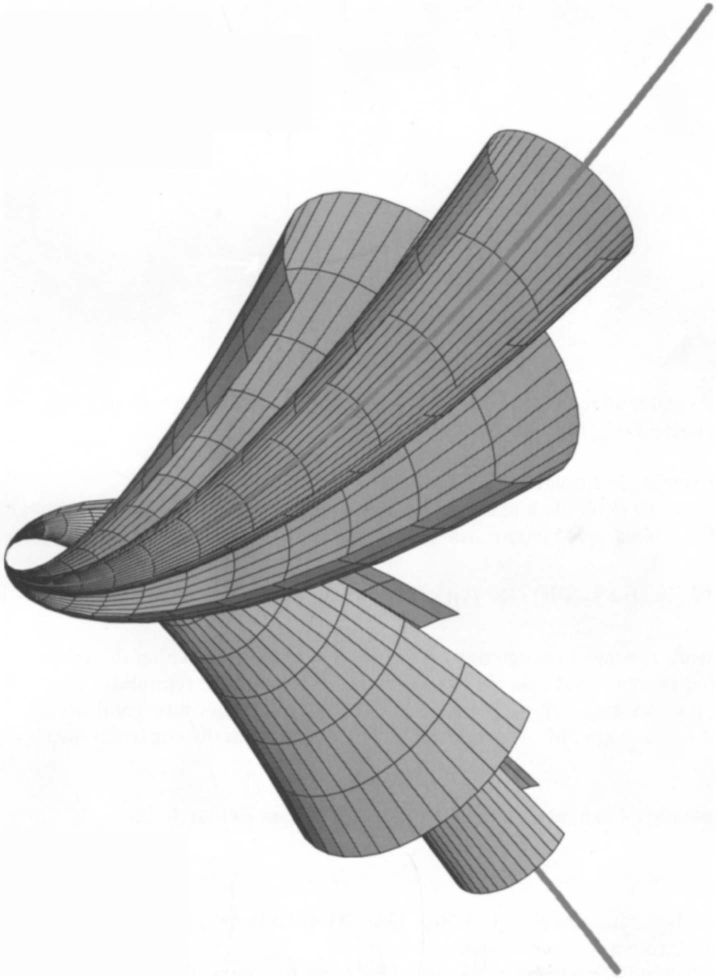

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Preface to the Second Edition

In July 1998, I received an e-mail from Alfred Gray, telling me:

> "... I am in Bilbao and working on the second edition of Tubes ... Tentatively, the new features of the book are:
>
> 1. Footnotes containing biographical information and portraits
> 2. A new chapter on mean-value theorems
> 3. A new appendix on plotting tubes
>
> ...".

That September he spent a week in Valencia, participating in a workshop on Differential Geometry and its Applications. Here he gave me a copy of the last version of Tubes. It could be considered a final version. There was only one point that we thought needed to be considered again, namely the possible completion of the material in Section 8.8 on comparison theorems of surfaces with the now well-known results in arbitrary dimensions. But only one month later the sad and shocking news arrived from Bilbao: Alfred had passed away. I was subsequently charged with the task of preparing the final revision of the book for the publishers, although some special circumstances prevented me from finishing the task earlier.

The book appears essentially as Alred Gray left it in September 1998. The only changes I carried out were the addition of Section 8.9 (representing the discussion we had), the inclusion of some new results on harmonic spaces, the structure of Hopf hypersurfaces in complex projective spaces and the conjecture about the volume of geodesic balls. Basic references associated with these additions have also been inserted. Other changes are just corrections of small mistakes and misprints.

The principal differences with respect to the first edition are those indicated in the e-mail reported above. The main subject of the book is the full understanding of Weyl's formula for the volume of a tube, its origins and its implications. Another approach to the study of volumes of tubes is the computation of the power series of the volume as a function of the tube radius, and this book also shows how this is accomplished using the method initiated in [Gr4]. The historical notes and the Mathematica plotting added to this second edition are inevitably valuable for a fuller appreciation of the mathematics. On the other hand, the chapter on

mean values, besides its intrinsic interest, demonstrates an interesting fact: techniques which are useful for volumes are also useful for problems related with the Laplacian.

There have been some important advances in the subject which are not included, since any attempt to incorporate them would have reduced the unity of the book and the clarity that is Alfred Gray's hallmark. I think that this book will remain, for a long time, compulsory reading for anyone who wants to be introduced to the subject.

I wish to thank Marisa Fernández and Mary Gray for their confidence in giving me the honor of carrying out the revision, and for their patience in waiting for the second edition to appear. Thanks are also due to Simon Salamon who helped check some linguistic (English and computer) aspects of the book. And my extreme gratitude to Alfred Gray, for his teaching and kind attention during the years I knew him.

Burjasot, January 2001
Vicente Miquel

Preface

This book is a revised version of notes for a course I gave during the spring of 1985 at the University of Santiago, Santiago de Compostela, Spain. My aim was to explain and to give some of the details of the proof of Weyl's Tube Formula and to discuss some generalizations. This formula (given in [Weyl1] has had a considerable influence, although indirect, on modern differential geometry through the theory of characteristic classes. Allendoerfer and Weil [AW] used Weyl's result to give the first proof of the generalized Gauss-Bonnet Theorem. Their proof is rather complicated. An intrinsic proof, which became much more popular because of its simplicity, was given by Chern in 1944 [Chern1]. Consequently, Weyl's original paper, although responsible for some of the key ideas in the proof of the Gauss-Bonnet Theorem, was put on the back shelf until recently. For example, the Selecta of Weyl's work [Weyl3] does not include Weyl's paper (although a volume published in the Soviet Union does contain Weyl's tube paper together with an interesting commentary by Arnol'd [Ar]). There are many important ideas in [Weyl1]; I hope that this small book will help to repopularize them.

Weyl's original proof of the tube formula used the classical differential geometry of Euclidean space. He also gave a formula for the volume of a tube about a submanifold of a sphere by using the standard embedding of a sphere in Euclidean space of one higher dimension and then transforming his tube formula for Euclidean space. It is now possible to give a simpler proof of Weyl's Tube Formula by making use of more modern techniques in differential geometry, for example, Jacobi vector fields. We prefer, however, to use the fact that the second fundamental forms of the tubular hypersurfaces satisfy a Riccati differential equation. This differential equation is equivalent to the Jacobi differential equation, and it can be argued that it has more geometric content because it gives direct information about the principal curvatures of the tubular hypersurfaces.

Using this Riccati differential equation for the second fundamental forms, it is possible to formulate a comparison theorem for the volume of a tube in a complete manifold of nonpositive or nonnegative sectional curvature (see Chapter 8). This comparison theorem has as a special case not only Weyl's Tube Formula, but also the comparison theorems of Bishop [Bishop] and Günther [Gü] for the volumes of geodesic balls.

For Kähler submanifolds of complex Euclidean space Weyl's Tube Formula simplifies enormously, because it is possible to express the coefficients in terms of Chern classes. In Theorem 7.7 we shall prove the **Complex Weyl Tube Formula**, namely that the volume $V_P^{\mathbb{C}^n}(r)$ of a tube of radius r about a complex submanifold P of $\mathbb{C}^n$ is given by

$$V_P^{\mathbb{C}^n}(r) = \frac{1}{n!}\int_P \gamma \wedge (\pi r^2 + F)^n.$$

Here P is assumed to have compact closure, γ denotes the total Chern form of P and F denotes the Kähler form of P. (See Chapter 6 for the relevant definitions.) There is also a tube formula (see Theorem 7.20) for complex submanifolds of complex projective space, which we call the **Projective Weyl Tube Formula**. Again the coefficients are expressed in terms of Chern classes. Although a complex projective space is more complicated than a sphere, the tube formulas for complex projective space are simpler than the corresponding formulas for a sphere.

A completely different approach to tube volumes is explained in Chapter 9: power series. First, a general method for computing the power series expansion for a geodesic ball in a general Riemannian manifold is discussed, and the classical formula of Bertrand-Diguet-Puiseux [BDP] is derived. A modification of this method yields the power series expansion of the volume of a tube of small radius r about a submanifold of a Riemannian manifold. A special case is the power series of Hotelling [Ht] for the volume of a tube about a curve in a Riemannian manifold. The papers [Weyl1] and [Ht] were published in the same issue of the American Journal of Mathematics as complements to one another. Recently, several authors have used and expanded on the ideas of [Weyl1] and [Ht] to study various problems in probability and statistics. Among them are [JS], [KS], [Nai1], [Nai2], [Smale], [GKP] and [GP].

Steiner's Formula is derived in Chapter 10 by the techniques used in earlier chapters for Weyl's Tube Formula. In fact, a tube about a connected orientable hypersurface of Euclidean space has two components. Steiner's Formula can be regarded as the formula for the volume of these half-tubes.

I wish to express my extreme gratitude to all my friends in Santiago de Compostela for making it possible to write these notes, and to the Korea Institute of Technology for the opportunity to lecture on this material at a conference during the summer of 1987. I wish to thank M. Elena Vázquez-Abal, Cornelia Bejan, Renzo Caddeo, Minshik Choi, Luis A. Cordero, Al Currier, Marisa Fernández, Mirjana Djorić, Pedro M. Gadea, William G. Goldman, Mary Gray, A.O. Ivanov, Alfredo Jiménez, Sungyun Lee, Vicente Miquel, Antonio M. Naveira, and A.A. Tuzhlin for their careful proof reading and many useful criticisms. Also, thanks go to Jan Beneš, Allan Wylde, Mona Zeftel and their team at Addison-Wesley for extensive help with the production of this book. Special thanks go to Garry Helzer for winning the copy of Mathematica that I used to do many of the illustrations. Finally, I wish to thank my many mathematical friends and colleagues with whom I have had fruitful discussions about tubes over the years. Among them, in addition

to the people mentioned above, are E. Abbena, Yu.D. Burago, D. Epstein, A.T. Fomenko, L. Geatti, P. Gilkey, L. Karp, O. Kowalski, M. Pinsky, R. Smith, L. Vanhecke and T.J. Willmore.

Chapter 1

An Introduction to Weyl's Tube Formula

The subject of this book is the computation and estimation of the volume of a tube about a submanifold of a Riemannian manifold. We explain some elementary aspects of the formula in Section 1.1. Before beginning the general theory, we carry out in Section 1.2 the computations in those low dimensional cases where the tubes can be easily visualized.

1.1 The Formula and Its History

In 1939 H. Weyl [Weyl1] derived a formula for the volume of a tube of small radius r around a submanifold P of dimension q in Euclidean space $\mathbb{R}^n$:

$$V_P^{\mathbb{R}^n}(r) = \frac{(\pi r^2)^{(n-q)/2}}{(\frac{1}{2}(n-q))!} \sum_{c=0}^{[q/2]} \frac{k_{2c}(P) r^{2c}}{(n-q+2)(n-q+4)\cdots(n-q+2c)}. \tag{1.1}$$

This formula has several noteworthy features. In the first place, the volume function is a polynomial in the radius r and not some more complicated function. Here

$$\frac{(\pi r^2)^{k/2}}{\left(\frac{k}{2}\right)!}$$

is a simple expression for the volume of the ball of radius r in $\mathbb{R}^k$. The coefficients $k_{2c}(P)$ are interesting functions; the first coefficient is just the volume of P.

Weyl observed that all the coefficients have a remarkable feature: they are independent of the particular way in which the submanifold P is embedded in

Euclidean space. He proved this fact by expressing the coefficients as integrals of certain rather complicated curvature functions. The coefficients have natural expressions in terms of the second fundamental form of the submanifold. So Weyl used his theory of invariants (see [Weyl2]) together with the Gauss equation to rewrite the coefficients $k_{2c}(P)$ in terms of the curvature of P. Thus, for example,

$$k_2(P) = \frac{1}{2} \int_P \tau \, dP,$$

where τ denotes the scalar curvature of P (see page 20). The other coefficients are integrals whose integrands are more complicated curvature functions. In formula (1.1) we assume that the relevant integrals converge (this is the case, for example, when P is compact) and that r is not too large. These assumptions, as well as the exact definition of a tube, will be made precise later. In the case when P is compact without boundary and even dimensional, say dim $P = 2p$, the top coefficient is especially interesting. We shall see in Chapter 5 that it is given by

$$k_{2p}(P) = (2\pi)^p \chi(P),$$

where $\chi(P)$ is the Euler characteristic of P.

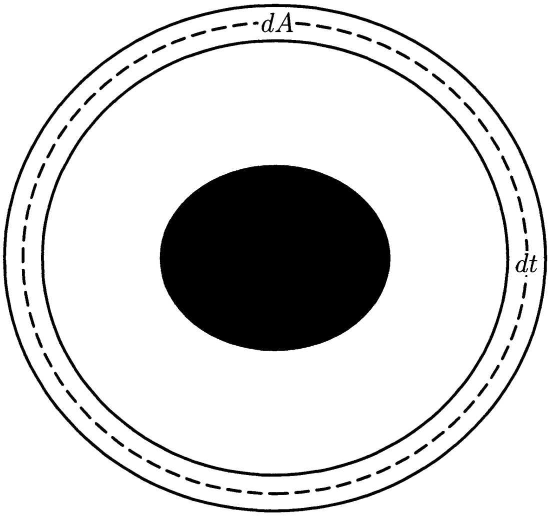
In Chapter 4 we shall give a proof of Weyl's Tube Formula using modern terminology; then we discuss generalizations. In 1940 Allendoerfer [Al1] and Fenchel [Fl2] combined Weyl's Tube Formula with H. Hopf's proof [Hopf1], [Hopf2] of the Gauss-Bonnet Theorem for compact hypersurfaces in an odd dimensional Euclidean space $\mathbb{R}^{2n+1}$. This yielded a simple elegant proof of the Gauss-Bonnet Theorem for compact submanifolds of arbitrary codimension in a Euclidean space (see Chapter 5). In Hopf's version of the Gauss-Bonnet Theorem the Euler characteristic of a compact hypersurface $P \subset \mathbb{R}^{2n+1}$ is expressed as an integral over P of the product of the principal curvatures of P; from it, it is hard to guess the Gauss-Bonnet formula for an abstract compact Riemannian manifold. Since the top coefficient $k_{2p}(P)$ in (1.1) is an integral over P of an integrand depending on the curvature of P, it is this integrand that suggests what form the Gauss-Bonnet Theorem should take for a general compact Riemannian manifold. In 1943 Allendoerfer and Weil [AW] used this idea; they gave a proof of the Gauss-Bonnet Theorem that consisted in patching together small pieces of a Riemannian manifold for which a local version of the Gauss-Bonnet Theorem was known to be true. In fact, they proved the Gauss-Bonnet Theorem for Riemannian manifolds with boundary.

1.2 Weyl's Formula for Low Dimensions

We shall give the general proof of the tube formula later in Chapter 4. However, it is instructive to work it out in those cases where the elementary theory of curves and surfaces can be applied. The relevant formulas can be found in almost any

differential geometry book that treats curves and surfaces, for example, in [ON1] or [Gr17]. One other fact will be needed:

$$V_P^{\mathbb{R}^n}(r) = \int_0^r \text{volume}(P_t)\,dt, \tag{1.2}$$



A cross section of a tube

where P_t denotes the tubular hypersurface at a distance t from P. This formula is almost obvious, as can be seen by examining a cross section of a tube using the methods of elementary calculus. (The volume element of the region between P_t and P is $dA\,dt$, where dA is the volume element of P_t.) Later, in Lemmas 3.13 (page 42) and 8.3 (page 147), we shall prove (1.2) in much more generality.

A Curve in $\mathbb{R}^2$

Recall that the **inner product** of vectors $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ in $\mathbb{R}^n$ is given by

$$\langle \mathbf{a}, \mathbf{b} \rangle = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n,$$

and we write $\|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$. We shall also need the **complex structure** of $\mathbb{R}^2$; it is the linear transformation $J \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$J(a_1, a_2) = (-a_2, a_1).$$

If $\mathbb{R}^2$ is identified with the complex numbers $\mathbb{C}$, then J is just multiplication by $\sqrt{-1}$.

Let $s \longmapsto \boldsymbol{\beta}(s)$ be a smooth curve in $\mathbb{R}^2$; we assume that $\boldsymbol{\beta}$ is defined for $a \le s \le b$ and has unit-speed, that is, $\|\boldsymbol{\beta}'(s)\| \equiv 1$. Then $\boldsymbol{\beta}$ has length $\text{Length}(\boldsymbol{\beta}) =$

$b - a$. Since both $\boldsymbol{\beta}''(s)$ and $J\boldsymbol{\beta}'(s)$ are perpendicular to $\boldsymbol{\beta}'(s)$, there exists a function $\boldsymbol{\kappa}_2$ such that

$$\boldsymbol{\beta}''(s) = \boldsymbol{\kappa}_2 J\boldsymbol{\beta}'(s).$$

The function $\boldsymbol{\kappa}_2$ is called the The **signed curvature** of $\boldsymbol{\beta}$; note that

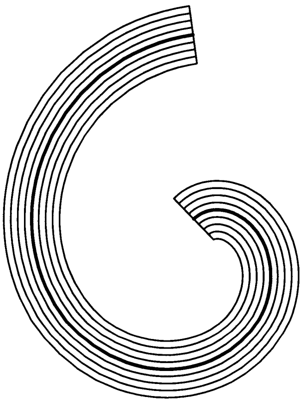
$$|\boldsymbol{\kappa}_2(s)| = \|\boldsymbol{\beta}''(s)\|.$$

Write $\mathbf{T} = \boldsymbol{\beta}'$ and $\mathbf{N} = J\boldsymbol{\beta}'$. We have the equations

$$\begin{cases} \mathbf{T}' = \boldsymbol{\kappa}_2 \mathbf{N}, \\ \mathbf{N}' = -\boldsymbol{\kappa}_2 \mathbf{T}. \end{cases} \tag{1.3}$$

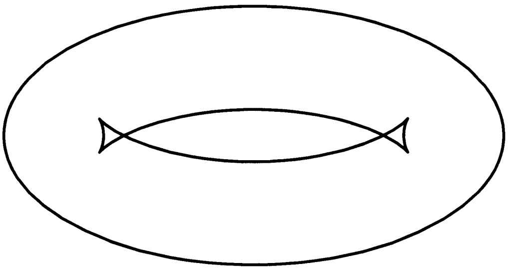
Put

$$\mathbf{X}_t(s) = \boldsymbol{\beta}(s) + t\,\mathbf{N}(s).$$



Curves parallel to
$t \longmapsto \big((t+1)\sin t, (t+1)\cos t\big)$

Then for small t the map $s \longmapsto \mathbf{X}_t(s)$ traces out a parallel curve at a distance $|t|$ from $\boldsymbol{\beta}$ (here t may be positive or negative). (If $|t|$ is too large, $s \longmapsto \mathbf{X}_t(s)$ may not be a regular curve; this happens, for example, when $\boldsymbol{\beta}$ is an ellipse. $s \longmapsto \mathbf{X}_t(s)$ may also have self intersections.)



Ellipse and a curve parallel to it

From (1.3) it follows that

$$\mathbf{X}_t'(s) = \mathbf{T}(s) + t\,\mathbf{N}'(s) = \big(1 - \boldsymbol{\kappa}_2(s)t\big)\mathbf{T}(s);$$

hence the length of $\mathbf{X}_t$ is

$$\mathrm{Length}(\mathbf{X}_t) = \int_a^b \|\mathbf{X}_t'(s)\|\,ds = \int_a^b \big(1 - \boldsymbol{\kappa}_2(s)t\big)ds,$$

provided the integrand is nonnegative. Thus for small $r \geq 0$ we have

$$\begin{aligned} V_{\boldsymbol{\beta}}^{\mathbb{R}^2}(r) &= \int_{-r}^{r} \mathrm{Length}(\mathbf{X}_t)\,dt = \int_a^b \int_{-r}^{r} (1 - \boldsymbol{\kappa}_2(s)t)\,dt\,ds \\ &= \int_a^b 2r\,ds = 2r\,\mathrm{Length}(\boldsymbol{\beta}). \end{aligned} \tag{1.4}$$

Notice that $V_{\boldsymbol{\beta}}^{\mathbb{R}^2}(r)$ depends only on r and the length of $\boldsymbol{\beta}$, and not on the curvature of $\boldsymbol{\beta}$.

A Curve in $\mathbb{R}^3$

Let

$$\mathbf{i} = (1,0,0) \qquad \mathbf{j} = (0,1,0) \qquad \mathbf{k} = (0,0,1).$$

be the standard unit vectors in $\mathbb{R}^3$. Recall that the **vector cross product** of vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$ is given by

$$\mathbf{a} \times \mathbf{b} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix},$$

or more explicitly,

$$\mathbf{a} \times \mathbf{b} = \det \begin{pmatrix} a_2 & a_3 \\ b_2 & b_3 \end{pmatrix} \mathbf{i} - \det \begin{pmatrix} a_1 & a_3 \\ b_1 & b_3 \end{pmatrix} \mathbf{j} + \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \mathbf{k}.$$

Suppose $s \longmapsto \boldsymbol{\beta}(s)$ is a unit-speed curve in $\mathbb{R}^3$. This time we use the familiar Frenet formulas (see, for example, [ON1, page 58] or [Gr17, page 186]):

$$\begin{cases} \mathbf{T}' = \boldsymbol{\kappa}\,\mathbf{N}, \\ \mathbf{N}' = -\boldsymbol{\kappa}\,\mathbf{T} + \boldsymbol{\tau}\,\mathbf{B}, \\ \mathbf{B}' = -\boldsymbol{\tau}\,\mathbf{N}, \end{cases} \tag{1.5}$$

where $s \longmapsto \{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ is the Frenet frame along $\boldsymbol{\beta}$, and $\boldsymbol{\kappa}$ and $\boldsymbol{\tau}$ denote the curvature and torsion of $\boldsymbol{\beta}$. The tubular surface at a distance t from $\boldsymbol{\beta}$ is described by means of the parametrization $(u, v) \longmapsto \mathbf{X}^t(u, v)$, where

$$\mathbf{X}^t(u, v) = \boldsymbol{\beta}(u) + t \cos v \, \mathbf{N}(u) + t \sin v \, \mathbf{B}(u). \tag{1.6}$$

We denote the partial derivatives of $\mathbf{X}^t$ with respect to u and v by $\mathbf{X}^t_u$ and $\mathbf{X}^t_v$. Then from (1.5) and (1.6) we have

$$\begin{aligned} \mathbf{X}^t_u(u, v) &= \big(1 - t\,\boldsymbol{\kappa}(u) \cos v\big)\mathbf{T}(u) - t\,\boldsymbol{\tau}(u) \sin v \, \mathbf{N}(u) + t\,\boldsymbol{\tau}(u) \cos v \, \mathbf{B}(u), \\ \mathbf{X}^t_v(u, v) &= -t \sin v \, \mathbf{N}(u) + t \cos v \, \mathbf{B}(u), \end{aligned}$$

so that the vector cross product of these two vector fields is given by

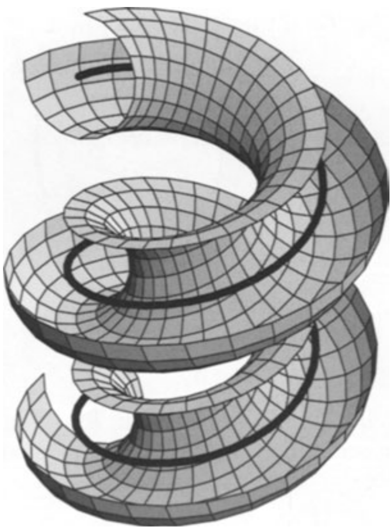
$$\begin{aligned} \mathbf{X}^t_u \times \mathbf{X}^t_v &= -t \sin v \big(1 - t\,\boldsymbol{\kappa}(u) \cos v\big)\mathbf{T}(u) \times \mathbf{N}(u) \\ &\qquad + t \cos v \big(1 - t\,\boldsymbol{\kappa}(u) \cos v\big)\mathbf{T}(u) \times \mathbf{B}(u) \\ &= -t \sin v \big(1 - t\,\boldsymbol{\kappa}(u) \cos v\big)\mathbf{B}(u) - t \cos v \big(1 - t\,\boldsymbol{\kappa}(u) \cos v\big)\mathbf{N}(u). \end{aligned}$$

Hence for small $t \geq 0$ we have

$$\|\mathbf{X}^t_u(u, v) \times \mathbf{X}^t_v(u, v)\| = t\big(1 - \boldsymbol{\kappa}(u) t \cos v\big),$$

and thus,

$$\int_0^{2\pi} \|\mathbf{X}^t_u(u, v) \times \mathbf{X}^t_v(u, v)\| dv = 2\pi\, t.$$



A tubular surface about the helix
$t \longmapsto (\cos t, \sin t, t/4)$

Therefore, we obtain

$$\text{volume}(P_t) = \int_a^b \int_0^{2\pi} \|\mathbf{X}_u^t \times \mathbf{X}_v^t\| dv\, du = \int_a^b 2\pi\, t\, du = 2\pi\, t\, \text{Length}(\boldsymbol{\beta}),$$

and so for small r

$$V_{\boldsymbol{\beta}}^{\mathbb{R}^3}(r) = \int_0^r \text{volume}(P_t)\, dt = \pi r^2 \text{Length}(\boldsymbol{\beta}).$$

Again, the tube volume depends only on the length of $\boldsymbol{\beta}$ and r.

A Surface in $\mathbb{R}^3$

Let $P \subset \mathbb{R}^3$ be a compact oriented surface. The orientation can be specified by the choice of a globally defined unit normal vector field on P. Denote by $P_{|t|}$ the tubular surface at a distance $|t|$ from P; it has two components $P_{|t|}^+$ and $P_{|t|}^-$.

We choose a local parametrization $\mathbf{z}\colon \mathcal{U} \longrightarrow P$ (sending (u, v) into $\mathbf{z}(u, v)$) for P. Also, let $\mathbf{U}\colon \mathcal{U} \longrightarrow \mathbb{R}^3$ be the function that assigns to each $(u, v) \in \mathcal{U}$ the unit vector normal to P at $\mathbf{z}(u, v)$; it is determined by the orientation of P. There is a small $\epsilon > 0$ such that for each t with $-\epsilon < t < \epsilon$ the map $\mathbf{Z}^t\colon \mathcal{U} \longrightarrow \mathbb{R}^3$ defined by

$$\mathbf{Z}^t(u, v) = \mathbf{z}(u, v) + t\,\mathbf{U}(u, v)$$

is injective. In fact, $P_{|t|}^+$ and $P_{|t|}^-$ can be chosen so that $\mathbf{Z}^t$ is a local parametrization for $P_{|t|}^+$ when $t > 0$, and a local parametrization for $P_{|t|}^-$ when $t < 0$. Note that

$$\mathbf{Z}_u^t = \mathbf{z}_u + t\,\mathbf{U}_u \qquad \text{and} \qquad \mathbf{Z}_v^t = \mathbf{z}_v + t\,\mathbf{U}_v. \tag{1.7}$$

Let S denote the shape operator of P; it is defined for surfaces by $Sv = -\nabla_v \mathbf{U}$ for any tangent vector v to P, where ∇ is the covariant derivative of $\mathbb{R}^3$. For more details see [ON1, page 190] or [Gr17, Chapter 16]. In particular,

$$S\mathbf{z}_u = -\mathbf{U}_u \qquad \text{and} \qquad S\mathbf{z}_v = -\mathbf{U}_v.$$

Therefore, (1.7) can be rewritten as

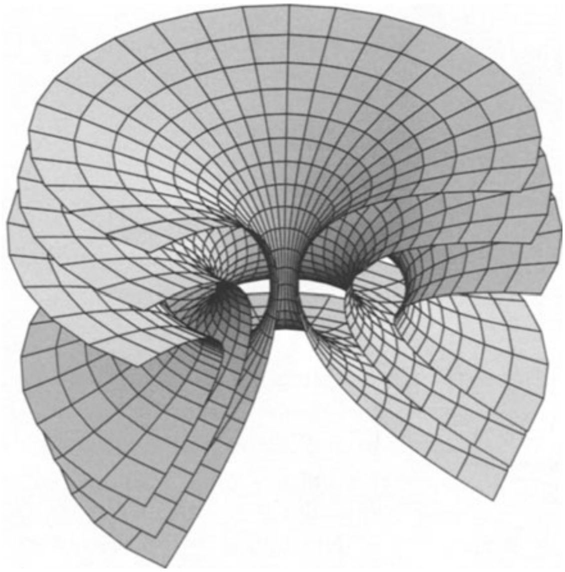
$$\mathbf{Z}_u^t = (I - t\,S)\mathbf{z}_u \quad \text{and} \quad \mathbf{Z}_v^t = (I - t\,S)\mathbf{z}_v, \tag{1.8}$$

and from (1.8) it follows that the vector cross product of $\mathbf{Z}_u^t$ and $\mathbf{Z}_v^t$ is given by

$$\mathbf{Z}_u^t \times \mathbf{Z}_v^t = \det(I - t\,S)\,\mathbf{z}_u \times \mathbf{z}_v.$$

The determinant is positive because t is small, and so

$$\|\mathbf{Z}_u^t \times \mathbf{Z}_v^t\| = \det(I - t\,S)\|\mathbf{z}_u \times \mathbf{z}_v\|.$$



Surfaces parallel to the catenoid
$(u,v) \longmapsto \big(\cos u \cosh v, \sin u \cosh v, v\big)$

On the other hand, there is a well-known formula expressing the eigenvalues of S in terms of H and K (see, for example, [ON1, page 203]):

$$\det(I - t\,S) = 1 - 2t\,H + t^2 K,$$

where H is the mean curvature and K is the Gaussian curvature of P. Then for a small $\epsilon > 0$ and $-\epsilon < t < \epsilon$ we have

$$\begin{aligned}
\operatorname{volume}(\mathbf{Z}^t(\mathcal{U})) &= \int_{\mathcal{U}} \|\mathbf{Z}^t_u \times \mathbf{Z}^t_v\|\,du\,dv \\
&= \int_{\mathcal{U}} (1 - 2t\,H + t^2 K)\|\mathbf{z}_u \times \mathbf{z}_v\|\,du\,dv \\
&= \operatorname{volume}(\mathbf{z}(\mathcal{U})) - 2t \int_{\mathcal{U}} H\|\mathbf{z}_u \times \mathbf{z}_v\|\,du\,dv \\
&\qquad + t^2 \int_{\mathcal{U}} K\|\mathbf{z}_u \times \mathbf{z}_v\|\,du\,dv \\
&= \operatorname{volume}(\mathbf{z}(\mathcal{U})) - 2t \int_{\mathbf{z}(\mathcal{U})} H\,dP + t^2 \int_{\mathbf{z}(\mathcal{U})} K\,dP.
\end{aligned}$$

Piecing together the local parametrizations, we obtain the formulas for the volume of $P_t^{\pm}$, namely

$$\mathrm{volume}(P_t^{\pm}) = \mathrm{volume}(P) \mp 2t \int_P H\, dP + t^2 \int_P K\, dP.$$

Hence

$$\mathrm{volume}(P_{|t|}) = \mathrm{volume}(P_t^{+}) + \mathrm{volume}(P_t^{-}) = 2\,\mathrm{volume}(P) + 2t^2 \int_P K\, dP.$$

Thus the tube formula is

$$\begin{aligned} V_P^{\mathbb{R}^3}(r) &= \int_0^r \mathrm{volume}(P_t)\, dt \\ &= 2r\ \mathrm{volume}(P) + \frac{2r^3}{3} \int_P K\, dP \end{aligned} \tag{1.9}$$

for small $r \geq 0$. This time the tube volume depends on r, volume(P) and $\int_P K\, dP$, but not on the shape operator.

The Gauss-Bonnet Theorem for compact surfaces (see, for example, [ON1, page 380]) states that

$$\int_P K\, dP = 2\pi\, \chi(P),$$

where $\chi(P)$ is the Euler characteristic of P. Hence, when P is compact, (1.9) can be rewritten as

$$V_P^{\mathbb{R}^3}(r) = 2r\ \mathrm{volume}(P) + \frac{4\pi r^3}{3}\chi(P).$$

Thus for a compact surface P, $V_P^{\mathbb{R}^3}(r)$ depends only on the volume and Euler characteristic of P.

Of course, tubes have many other uses that will not be discussed here. We mention only Fenchel's Theorem [Fl1]: *The total curvature of a simple closed curve in* $\mathbb{R}^3$ *is greater or equal to* 2π. *Equality holds if and only if the curve is a plane convex curve.* For a proof of this theorem using the elementary tube techniques of this chapter, as well as the theorem of Fary [Fary] and Milnor [Mil1] which generalizes it, see [dC, page 399].

A Ball in $\mathbb{R}^n$

When P is a point $m \in \mathbb{R}^n$, the tube formula reduces to the formula for the volume of a geodesic ball in $\mathbb{R}^n$:

$$V_m^{\mathbb{R}^n}(r) = \frac{(\pi r^2)^{n/2}}{\left(\frac{n}{2}\right)!}. \tag{1.10}$$

For a proof of this formula see Lemma 1.4 of the Appendix (page 248). In Chapter 9 we shall find a power series expansion that is a generalization of (1.10) to an arbitrary Riemannian manifold.

Half-Tubes and Steiner's Formula

Let us return to the computation of the volume of a tube about a curve $u \longmapsto \beta(u)$ in $\mathbb{R}^2$. A tube of radius r about $u \longmapsto \beta(u)$ would in everyday language be called a strip of width $2r$. Moreover, such a strip is naturally divided into two parts, an upper half and a lower half. These two halves have volumes, which we denote by $V_\beta^{\mathbb{R}^2+}(r)$ and $V_\beta^{\mathbb{R}^2-}(r)$. It is clear that

$$\mathrm{V}_\beta^{\mathbb{R}^{2+}}(r) + \mathrm{V}_\beta^{\mathbb{R}^{2-}}(r) = \mathrm{V}_\beta^{\mathbb{R}^2}(r).$$

In fact, it is no more difficult to compute $V_\beta^{\mathbb{R}^2\pm}(r)$ than it is to compute $V_\beta^{\mathbb{R}^2}(r)$. Using essentially the same calculations as those yielding (1.4), we find that

$$\begin{aligned} \mathrm{V}_\beta^{\mathbb{R}^{2\pm}}(r) &= \int_0^r L(\mathbf{X}_t)\,dt = \int_a^b \int_0^r \big(1 \mp \boldsymbol{\kappa}_2(u)t\big)\,dt\,du \qquad (1.11) \\ &= \int_a^b \left(r \mp \frac{r^2}{2}\boldsymbol{\kappa}_2(u)\right)du \\ &= r\,L(\beta) \mp \frac{r^2}{2}\int_a^b \boldsymbol{\kappa}_2(u)\,du. \end{aligned}$$

Notice that there is an essential difference between the volume of a half-tube and the volume of a tube about a curve in $\mathbb{R}^2$: the volume of the half-tube depends on the curvature of the curve, whereas the volume of the full tube is independent of it. It is remarkable that when the volumes of the two half-tubes are summed, the dependency on curvature disappears. In Chapter 10 we shall see that this phenomenon also occurs in higher dimensions for hypersurfaces.

The calculation of the volumes of half-tubes is related to a formula that the great Swiss geometer Steiner proved in 1840 (see [Sr]). Let B be a compact subset of $\mathbb{R}^2$. Steiner assumed that B was convex with piecewise linear boundary, but we shall assume that B is possibly nonconvex but has smooth boundary ∂B. At any rate, the final results are almost the same.[1] Steiner computed the volume of the set B_r of points within a distance r of B.

It is clear that the volume of B_r is equal to the sum of the volume of B and the outward half-tube emanating from the boundary ∂B. Hence, from (1.11) we get

$$\mathrm{Area}(B_r) = \mathrm{Area}(B) + \mathrm{Length}(\partial B)r - \frac{r^2}{2}\int_{\partial B}\boldsymbol{\kappa}_2(u)\,du. \qquad (1.12)$$

When the Gauss-Bonnet Theorem is applied to the flat region B, we see that

$$\int_{\partial B}\boldsymbol{\kappa}_2(u)\,du = -2\pi\chi(B), \qquad (1.13)$$

[1] But difficulties arise if one tries to do the case when B is neither convex nor has smooth boundary, because there may be overlapping for arbitrarily small r at the corners.

where $\chi(B)$ denotes the Euler characteristic of B. Then (1.12) can be rewritten using (1.13) as

$$\mathrm{Area}(B_r) = \mathrm{Area}(B) + \mathrm{Length}(\partial B)r + \pi\chi(B)r^2.$$

Similarly, one can consider half-tubes about orientable hypersurfaces of $\mathbb{R}^n$ and compute their volumes. For example, when $n = 3$ formula (1.9) can be refined to a formula for the volume of a half-tube about a surface $P \subset \mathbb{R}^3$. For $r > 0$ let $V_P^{\mathbb{R}^3 \pm}(r)$ denote the portion of the tube about P that lies between P and $P_r^{\pm}$. Then the refined version of (1.9) is

$$\begin{aligned} V_P^{\mathbb{R}^3 \pm}(r) &= \int_0^r \mathrm{volume}(P_t^{\pm})\, dt \\ &= r\ \mathrm{volume}(P) \mp r^2 \int_P H\, dP + \frac{r^3}{3}\int_P K\, dP. \end{aligned}$$

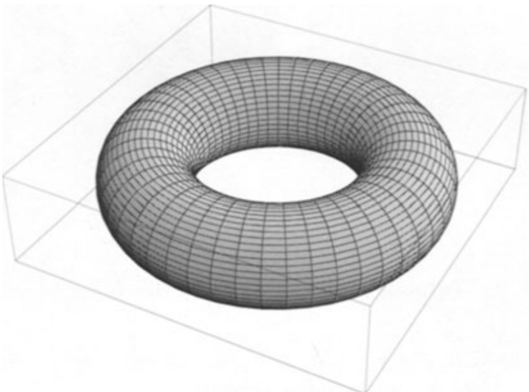
We shall return to this theme in Chapter 10.

1.3 Problems

1.1 Compute by hand the volume of a torus in $\mathbb{R}^3$ parametrized by

$$\mathbf{torus[a,b]}(u,v) = \big((a + b\cos v)\cos u, (a + b\cos v)\sin u, b\sin v\big),$$

where $a > b > 0$. Then compute the volume using Weyl's formula and compare the results.



torus[8,3]

1.2 Embed $\mathbf{torus[a,b]}$ in $\mathbb{R}^n$ in the natural way, where $n \geq 3$. Use Weyl's tube formula to compute the volume of a tube about $\mathbf{torus[a,b]}$ considered as a submanifold of $\mathbb{R}^n$.

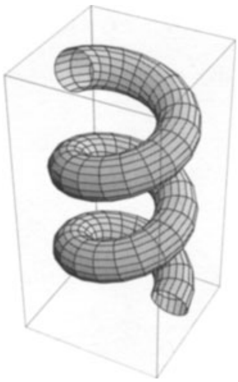
1.3 Consider the helix in $\mathbb{R}^3$ parametrized by

$$\mathbf{helix[a,b]}(t) = (a\cos t, a\sin t, bt).$$

$\mathbf{helix[1,0.3]}$ $(0 \le t \le 5\pi)$

a. Find the length of $\mathbf{helix[a,b]}(t)$ for $t_1 \le t \le t_2$.

b. Find the parametrization of a tube of radius r about $\mathbf{helix[a,b]}(t)$.



Tube of radius 0.4 about the helix helix[1,0.3]

c. Compute the volume of a tube of radius r about $\mathbf{helix[a,b]}(t)$ both by hand and using Weyl's tube formula.

Chapter 2

Fermi Coordinates and Fermi Fields

In this chapter we shall be concerned with the geometry of tubes about a submanifold P of a general Riemannian manifold M (and not specifically tubes in Euclidean space). In Section 2.1 we define and discuss normal and Fermi coordinates. Section 2.2 is devoted to a quick review of the curvature tensor of a Riemannian manifold and its various contractions. Instead of working directly with Fermi coordinates, it is usually easier to use certain vector fields, which we call Fermi fields and define in Section 2.3. There is a close relation between Fermi fields and the more familiar Jacobi fields (see Corollaries 2.9 and 2.10). In Chapter 3 we shall derive three fundamental equations to describe the geometry of tubes using Fermi fields. Since Fermi coordinates are a generalization of normal coordinates, it is not surprising that there is a tube generalization of the well-known Gauss Lemma; this we prove in Section 2.4.

2.1 Fermi Coordinates as Generalized Normal Coordinates

The proof of Weyl's Tube Formula (1.1) requires a good understanding of the geometry of a Riemannian manifold M in a neighborhood of a submanifold P. The most convenient coordinates to use for this are Fermi[1] coordinates. A comprehensive discussion of Fermi coordinates about a curve in a surface has been given by Fiala [Fiala, Section 6]. See also [Da, volume 2, pages 438–440], and [Har].

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[1] Enrico Fermi (1901–1954). The Fermi of Fermi coordinates is the same Fermi who later became more famous as a physicist. He won the Nobel prize in 1938. His papers [Fermi] treat the case when P is a curve.

Fermi coordinates are a generalization of normal coordinates, which are probably more familiar to most readers. It turns out that many facts about normal coordinates (such as the Gauss Lemma) have generalizations to Fermi coordinates. Let us recall the properties of normal coordinates as an aid to understanding Fermi coordinates, which are somewhat more complicated. Normal coordinates are the natural coordinates to use in the study of a geodesic ball, which is a simple but important special case of a tube.

Let M be a Riemannian manifold. (Unless otherwise stated all manifolds and maps will be of class C^∞, and all manifolds will be paracompact.) We denote by M_m the tangent space to M at m. For $v \in M_m$ let ξ_v denote the unique geodesic in M with $\xi_v(0) = m$ and $\xi_v'(0) = v$. We write $\exp_m(v) = \xi_v(1)$, provided that $\xi_v(t)$ can be defined for $t = 1$. This is the **exponential**[2] **map** of M at m; although it may be defined only on a neighborhood of $0 \in M_m$, we shall write $\exp_m\colon M_m \longrightarrow M$.

Each tangent space M_m is a differentiable manifold of the same dimension as M. Let $(M_m)_0$ denote the tangent space to M_m at the origin. It is easy to prove (see problem 2.1) that the tangent map to $\exp_m$ at $0 \in M_m$ is the canonical identification between M_m and its tangent space $(M_m)_0$ at 0. Hence by the inverse function theorem $\exp_m$ is a diffeomorphism in a neighborhood of $0 \in M_m$.

Definition. *Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of M_m. For $1 \le j \le n$ define a real-valued function x_j on a neighborhood of m by*

$$x_j\left(\exp_m\left(\sum_{i=1}^n t_i e_i\right)\right) = t_j.$$

Then $(x_1, \ldots, x_n)$ is the system of **normal coordinates** *corresponding to the orthonormal basis $\{e_1, \ldots, e_n\}$.*

Next we use a system of normal coordinates to define some special vector fields. Let

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$$

be the coordinate vector fields associated with the normal coordinate system $(x_1, \ldots, x_n)$.

[2]The reason for the name "exponential map" is briefly as follows. The ordinary exponential function from real and complex variables, that is,

$$z \longmapsto e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!}, \tag{2.1}$$

can be extended to square matrices, because the right-hand side of (2.1) makes sense when z is replaced by a square matrix A. This new map $A \longmapsto e^A$ provides a method of mapping the vector space of all $n \times n$ matrices onto the group of all nonsingular $n \times n$ matrices. In particular, it maps the vector space of all antisymmetric $n \times n$ matrices onto the group $\mathbf{SO}(n)$ of all orthogonal $n \times n$ matrices of determinant 1. It turns out that there is a natural Riemannian metric on $\mathbf{SO}(n)$ for which $t \longmapsto e^{tA}$ is a geodesic in $\mathbf{SO}(n)$ for any $n \times n$ antisymmetric matrix A. This fact can be generalized to any Riemannian manifold; the result is exp.

Definition. *A* **normal coordinate vector field** *at m is a vector field X (defined in a neighborhood of $m \in M$) of the form*

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i},$$

where the a_i's are constants.

For many purposes it turns out to be far easier to work with normal coordinate vector fields instead of normal coordinates. Notice, for example, that:

Lemma 2.1. *The notion of normal coordinate vector field at m does not depend on the choice of normal coordinate system at m.*

Proof. Any system of normal coordinates at m can be rotated into another system by means of a *constant* orthogonal matrix. □

Fermi coordinates are the natural generalization of normal coordinates that arises when one replaces the point m by a submanifold P. For the moment we assume only that P is a topologically embedded submanifold of M, although ultimately we shall be interested mainly in the case when the closure of P is compact. Let $\boldsymbol{\nu}$ denote the **normal bundle** of P in M. Thus

$$\boldsymbol{\nu} = \{\, (p, v) \mid p \in P \ \text{ and } \ v \in P_p^{\perp} \,\},$$

where $P_p^{\perp}$ denotes the orthogonal complement of P_p in M_p. Then $\boldsymbol{\nu}$ is a vector bundle over P, and so a differentiable manifold. (In fact, $\boldsymbol{\nu}$ is a subbundle of the restriction to P of the tangent bundle of M.) The **exponential map of the normal bundle $\boldsymbol{\nu}$** is the map $\exp_\nu$ defined by

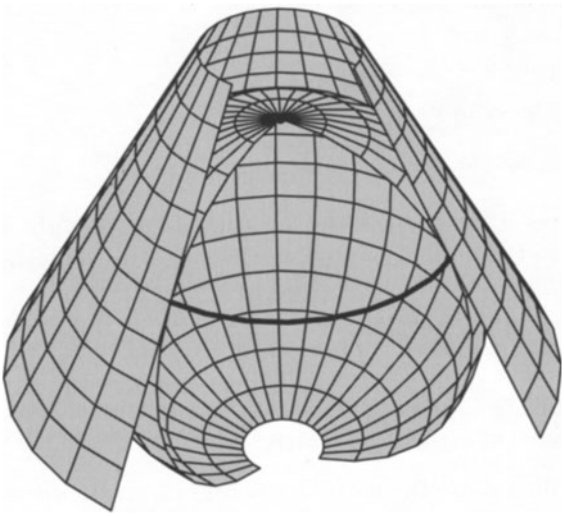
$$\exp_\nu(p, v) = \exp_p(v)$$

for $(p, v) \in \boldsymbol{\nu}$. (Here $\exp_p$ denotes the exponential map of M at p.) Thus

$$\exp_\nu \colon \boldsymbol{\nu} \longrightarrow M,$$

although strictly speaking $\exp_\nu$ may be defined only in a neighborhood of the zero section of $\boldsymbol{\nu}$. The generalized Gauss Lemma of Section 2.4 will give the details of how $\exp_\nu$ distorts distances. We can identify P with the zero section of $\boldsymbol{\nu}$, so that P can be regarded as a submanifold of $\boldsymbol{\nu}$ as well as a submanifold of M. This implies that for each $p \in P$ we have the inclusion $P_p \subseteq \boldsymbol{\nu}_{(p,0)}$, where $\boldsymbol{\nu}_{(p,0)}$ denotes the tangent space to $\boldsymbol{\nu}$ at $(p, 0)$. Moreover, from the definition of $\boldsymbol{\nu}$ we also have that $P_p^{\perp} \subseteq \boldsymbol{\nu}_{(p,0)}$; with these identifications it is clear that

$$\boldsymbol{\nu}_{(p,0)} = P_p \oplus P_p^{\perp}.$$



The normal bundle of a nonequatorial circle on a sphere is a cone

It is possible to define in a natural way a Riemannian metric on $\boldsymbol{\nu}$ (see problem 2.2). This is a special case of the construction of a metric on the total space of a Riemannian submersion.

The differential of $\exp_\nu$ becomes very simple when restricted to P.

Lemma 2.2. *Let $p \in P$. Then:*

(i) *the restriction of $\big((\exp_\nu)_*\big)_{(p,0)}$ to $(P_p^\perp)_0$ is the canonical identification of $(P_p^\perp)_0$ with $P_p^\perp$;*

(ii) *the restriction of $\big((\exp_\nu)_*\big)_{(p,0)}$ to $(P_p)_0$ is the canonical inclusion of P_p in M_p.*

Proof. Part (i) is a consequence of the fact that the tangent map of the exponential map of M at the origin is the canonical identification between $(M_p)_0$ and M_p (see problem 2.1). Also, since the restriction of $\exp_\nu$ to the zero section is just the inclusion of P in M, we get (ii) by taking the tangent maps. □

Lemma 2.3. *Let P be a topologically embedded submanifold of a Riemannian manifold M. Then the map $\exp_\nu\colon \boldsymbol{\nu} \longrightarrow M$ maps a neighborhood of $P \subset \boldsymbol{\nu}$ diffeomorphically onto a neighborhood of $P \subset M$.*

Proof. It follows from Lemma 2.2 that $\big((\exp_\nu)_*\big)_{(p,0)}\colon \boldsymbol{\nu}_{(p,0)} \longrightarrow M_p$ is an isomorphism for each $p \in P$. Then the inverse function theorem implies that for each

$p \in P$, $\exp_\nu$ maps a neighborhood of $(p, 0) \in \boldsymbol{\nu}$ diffeomorphically onto a neighborhood of $p \in M$. For the proof that the union of these neighborhoods contains a neighborhood that is mapped diffeomorphically by $\exp_\nu$ onto a neighborhood of $P \subset M$ see [ON4, page 200]. $\square$

Let $\mathcal{O}_P$ be the subset of $\boldsymbol{\nu}$ defined by

$$\begin{aligned} \mathcal{O}_P \;=\; & \text{the largest neighborhood of the zero section of } \boldsymbol{\nu} \text{ for} \\ & \text{which } \exp_\nu \colon \mathcal{O}_P \longrightarrow \exp_\nu(\mathcal{O}_P) \text{ is a diffeomorphism.} \end{aligned} \tag{2.2}$$

To define a system of Fermi coordinates, we need an arbitrary system of coordinates $(y_1, \ldots, y_q)$ defined in a neighborhood $\mathcal{V} \subset P$ of $p \in P$ together with orthonormal sections $E_{q+1}, \ldots, E_n$ of the restriction of $\boldsymbol{\nu}$ to $\mathcal{V}$.

Definition. *The* **Fermi coordinates** $(x_1, \ldots, x_n)$ *of* $P \subset M$ *centered at* p *(relative to a given coordinate system* $(y_1, \ldots, y_q)$ *on* P *and given orthonormal sections* $E_{q+1}, \ldots, E_n$ *of* $\boldsymbol{\nu}$*) are defined by*

$$x_a\left(\exp_\nu\left(\sum_{j=q+1}^{n} t_j E_j(p')\right)\right) = y_a(p') \qquad (a = 1, \ldots, q), \tag{2.3}$$

$$x_i\left(\exp_\nu\left(\sum_{j=q+1}^{n} t_j E_j(p')\right)\right) = t_i \qquad (i = q+1, \ldots, n), \tag{2.4}$$

for $p' \in \mathcal{V}$, *provided the numbers* $t_{q+1}, \ldots, t_n$ *are small enough so that*

$$\sum_{j=q+1}^{n} t_j E_j(p') \in \mathcal{O}_P.$$

In what follows we make the following notational conventions:

the x_a $(1 \le a \le q)$ are given by (2.3),

the x_i $(q + 1 \le i \le n)$ are given by (2.4).

Since $\exp_\nu$ is a diffeomorphism on $\mathcal{O}_P$, equations (2.3) and (2.4) actually define a coordinate system near p.

Fermi coordinates measure the geometry of a Riemannian manifold M in a neighborhood of a submanifold P. Therefore, we are principally interested in how the x_a and the x_i vary along geodesics normal to P. Hence the choice of the coordinate system on P is not important; if necessary it can be chosen to be a system of normal coordinates.

The next two elementary lemmas will be needed in Section 2.3.

Lemma 2.4. *If $(x_1, \ldots, x_n)$ is a system of Fermi coordinates centered at $p \in P$, then the restrictions to P of the coordinate vector fields*

$$\frac{\partial}{\partial x_{q+1}}, \ldots, \frac{\partial}{\partial x_n}$$

are orthonormal.

Proof. Let $p' \in P$ be near p. For $q+1 \leq i \leq n$ the integral curve of $\dfrac{\partial}{\partial x_i}$ starting at p' is the geodesic ξ defined by $\xi(t) = \exp_\nu\big(p', tE_i(p')\big)$. So

$$\left.\frac{\partial}{\partial x_i}\right|_{p'} = \xi'(0) = E_i(p').$$

Since the $E_i(p')$'s are orthonormal, the result follows. $\square$

Let

$$\delta_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

Lemma 2.5. *Let ξ be a unit-speed geodesic normal to P with $\xi(0) = p \in P$, and let $u = \xi'(0)$. Then there is a system of Fermi coordinates $(x_1, \ldots, x_n)$ such that for small t (that is, for $(p, tu) \in \mathcal{O}_P$) we have*

$$\left.\frac{\partial}{\partial x_{q+1}}\right|_{\xi(t)} = \xi'(t), \tag{2.5}$$

and

$$\left.\frac{\partial}{\partial x_a}\right|_p \in P_p, \qquad \left.\frac{\partial}{\partial x_i}\right|_p \in P_p^\perp \tag{2.6}$$

for $1 \leq a \leq q$ and $q+1 \leq i \leq n$. Furthermore,

$$(x_\alpha \circ \xi)(t) = t\delta_{\alpha q+1} \tag{2.7}$$

for $1 \leq \alpha \leq n$.

Proof. Choose an orthonormal frame $\{e_1, \ldots, e_n\}$ at p so that $\{e_1, \ldots, e_q\}$ is a basis of P_p and $e_{q+1} = \xi'(0)$. Extend $e_{q+1}, \ldots, e_n$ to orthonormal sections $E_{q+1}, \ldots, E_n$ of $\boldsymbol{\nu}$ in a neighborhood of p, and let $(y_1, \ldots, y_q)$ be the normal coordinates on P defined by $e_1, \ldots, e_q$. Then $(y_1, \ldots, y_q)$ and the sections $E_{q+1}, \ldots, E_n$ give rise to a system $(x_1, \ldots, x_n)$ of Fermi coordinates centered at p that clearly satisfies (2.6). Since the integral curve of $\dfrac{\partial}{\partial x_{q+1}}$ starting at p is a geodesic with the same initial velocity as ξ, we get (2.5). For any coordinate system $(x_1, \ldots, x_n)$ we have the formula

$$\xi'(t) = \sum_{\alpha=1}^n (x_\alpha \circ \xi)'(t) \left.\frac{\partial}{\partial x_\alpha}\right|_{\xi(t)}. \tag{2.8}$$

Then (2.5) and (2.8) imply (2.7). $\square$

2.2 A Review of Curvature Fundamentals

Let M be any C^∞ differentiable manifold, and denote by $\mathfrak{F}(M)$ the algebra of C^∞ functions on M. As usual, we let $\mathfrak{X}(M)$ be the Lie algebra of C^∞ vector fields on M; it can be taken to be the derivation algebra of the algebra $\mathfrak{F}(M)$. A **metric tensor** on M is a symmetric $\mathfrak{F}(M)$–bilinear form $\langle\ ,\ \rangle$ on $\mathfrak{X}(M)$ with values in $\mathfrak{F}(M)$. It gives rise to an inner product (which will also be denoted by $\langle\ ,\ \rangle$) on each tangent space M_m. Although we shall assume that $\langle\ ,\ \rangle$ is positive definite, many of the theorems that we state hold also in the indefinite case. A **Riemannian manifold** consists of a differentiable manifold M equipped with a metric tensor $\langle\ ,\ \rangle$.

The **Riemannian connection** (or **covariant derivative**) ∇ of a Riemannian manifold M is defined by

$$\begin{aligned} 2\langle\nabla_X Y, Z\rangle &= X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle \\ &\qquad - \langle X, [Y, Z]\rangle - \langle Y, [X, Z]\rangle + \langle Z, [X, Y]\rangle \end{aligned} \tag{2.9}$$

for $X, Y, Z \in \mathfrak{X}(M)$. Then ∇ has the following properties and is characterized by them:

$$\nabla_{fX+gY} = f\nabla_X + g\nabla_Y, \tag{2.10}$$

$$\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z, \tag{2.11}$$

$$\nabla_X fY = (Xf)Y + f\nabla_X Y, \tag{2.12}$$

$$\nabla_X Y - \nabla_Y X = [X, Y], \tag{2.13}$$

$$X\langle Y, Z\rangle = \langle\nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \tag{2.14}$$

for $X, Y, Z \in \mathfrak{X}(M)$ and $f, g \in \mathfrak{F}(M)$. Notice that ∇ is not a tensor field, because it is not linear with respect to functions in its second argument.

The **curvature transformation** of M is defined by

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y] \tag{2.15}$$

for $X, Y \in \mathfrak{X}(M)$. It satisfies the following identities:

$$R_{XY} = -R_{YX}, \tag{2.16}$$

$$\langle R_{WX}Y, Z\rangle = -\langle R_{WX}Z, Y\rangle, \tag{2.17}$$

$$\langle R_{WX}Y, Z\rangle = \langle R_{YZ}W, X\rangle, \tag{2.18}$$

$$\underset{XYZ}{\mathfrak{S}} R_{XY}Z = 0, \tag{2.19}$$

for $W, X, Y, Z \in \mathfrak{X}(M)$. Here $\mathfrak{S}$ denotes the cyclic sum. Equation (2.19) is usually called the **first Bianchi identity**. We often write

$$R_{WXYZ} = \langle R_{WX}Y, Z\rangle;$$

this is the notation for the **curvature tensor**. The name is justified because in contrast to the covariant derivative, the curvature tensor is linear with respect to functions in all its arguments.

Algebraically the curvature tensor is a rather complicated object, so often it is important to study its contractions. There are two such contractions; to define them, let $\{E_1, \ldots, E_n\}$ be an orthonormal frame field defined on an open subset of M. $m \in M$.) Then the **Ricci curvature** ρ is given by

$$\rho(X, Y) = \sum_{a=1}^{n} R_{XE_aYE_a}$$

for $X, Y \in \mathfrak{X}(M)$, where n is the dimension of M. Because of (2.18) the Ricci curvature ρ is symmetric in X and Y. Similarly, the **scalar curvature** τ is defined by

$$\tau = \sum_{a=1}^{n} \rho(E_a, E_a) = \sum_{ab=1}^{n} R_{E_aE_bE_aE_b}.$$

It is easily verified that these definitions do not depend on the choice of local orthonormal frame field $\{E_1, \ldots, E_n\}$. (A proof can be given along the lines of Lemma 2.6 of the next section.)

The **sectional curvature** is the function K of vector fields X and Y given by

$$K_{XY} = \frac{R_{XYXY}}{\|X\|^2\|Y\|^2 - \langle X, Y\rangle^2}.$$

It is defined wherever X and Y are linearly independent. An easy argument shows that at each point $p \in M$ the value of K_{XY} depends only on the two dimensional subspace Π_p (or **section**) spanned by the vector fields X and Y at p. As a result we write $K(\Pi_p) = K_{XY}(p)$. In particular, when M is two dimensional, $\Pi_p = M_p$, so there is only one sectional curvature. It is usually called the Gaussian curvature and simply denoted by K.

There are many details about curvature that we have not covered in this section, such as the second Bianchi identity and the quadratic invariants of the curvature. We shall introduce them when we need them.

2.3 Fermi Fields

Calculations involving Fermi coordinates are considerably simplified by the use of **Fermi vector fields**, which are the analogs for Fermi coordinates of normal

coordinate vector fields. If P is a topologically embedded submanifold of M and $p \in P$, let $\mathcal{U} \subseteq \mathcal{O}_P$ on which a system of Fermi coordinates $(x_1, \ldots, x_n)$ centered at p is defined. As usual $\mathfrak{X}(\mathcal{U})$ denotes the Lie algebra of C^∞ vector fields on $\mathcal{U}$.

Definition. *We say that $A \in \mathfrak{X}(\mathcal{U})$ is a* **tangential Fermi field** *provided*

$$A = \sum_{a=1}^{q} c_a \frac{\partial}{\partial x_a},$$

where the c_a's are constants. Similarly, a vector field $X \in \mathfrak{X}(\mathcal{U})$ of the form

$$X = \sum_{i=q+1}^{n} d_i \frac{\partial}{\partial x_i},$$

where the d_i's are constants, is called a **normal Fermi field**.

For $p \in P$ we denote by $\mathfrak{X}(P,p)^\top$ and $\mathfrak{X}(P,p)^\perp$ the spaces of tangential and normal Fermi fields at $p \in P$. It is easy to see that they are both vector spaces and have dimensions q and $n-q$, respectively. Also, let

$$\mathfrak{X}(P,p) = \mathfrak{X}(P,p)^\top \oplus \mathfrak{X}(P,p)^\perp$$

be the space of Fermi fields at p. When P is a point, the normal Fermi fields coincide with the normal coordinate fields defined above. Notice that if $E_{q+1}, \ldots, E_n$ are rotated by a constant orthogonal matrix, the space of Fermi fields remains the same.

So instead of Fermi coordinates we prefer to use Fermi fields; this technique avoids a great deal of complicated notation. Using Fermi fields, we shall show very simply in Chapter 3 that the second fundamental forms of tubular hypersurfaces satisfy a Riccati differential equation.

Two other simple objects, σ and N, will be needed for the study of tubes. We define them in terms of Fermi coordinates and then describe them geometrically.

Definition. *Let $(x_1, \ldots, x_n)$ be a system of Fermi coordinates for $P \subset M$. For $\sigma > 0$ we put*

$$\sigma^2 = \sum_{i=q+1}^{n} x_i^2 \quad \textit{and} \quad N = \sum_{i=q+1}^{n} \frac{x_i}{\sigma} \frac{\partial}{\partial x_i}. \tag{2.20}$$

For geometric interpretations we need some properties of σ and N.

Lemma 2.6. *Let $p \in P$. Then the definitions of σ and N are independent of the choice of Fermi coordinates at p.*

Proof. Let $(x'_1, \dots, x'_n)$ be another system of Fermi coordinates centered at p, and let $\{E'_{q+1}, \dots, E'_n\}$ be the orthonormal sections of ν that give rise to it. We can write

$$E'_k = \sum_{j=q+1}^{n} a_{jk} E_j,$$

where (a_{jk}) is a matrix of functions in the orthogonal group $O(n-q)$ with each $a_{jk} \in \mathfrak{F}(P)$. Then

$$\begin{aligned} x_i\left(\exp_\nu\left(\sum_{k=q+1}^{n} t'_k E'_k\right)\right) &= x_i\left(\exp_\nu\left(\sum_{j=q+1}^{n}\left(\sum_{k=q+1}^{n} a_{jk} t'_k\right) E_j\right)\right) \\ &= \sum_{l=q+1}^{n} a_{il} t'_l \\ &= \sum_{l=q+1}^{n} a_{il} x'_l\left(\exp_\nu\left(\sum_{k=q+1}^{n} t'_k E'_k\right)\right), \end{aligned}$$

from which we conclude that $x_i = \sum_{j=q+1}^{n} a_{ij} x'_j$. Then

$$\begin{aligned} \sum_{i=q+1}^{n} x_i^2 &= \sum_{i=q+1}^{n} \sum_{j=q+1}^{n} \sum_{k=q+1}^{n} a_{ij} x'_j a_{ik} x'_k \\ &= \sum_{j=q+1}^{n} \sum_{k=q+1}^{n} \left(\sum_{i=q+1}^{n} a_{ij} a_{ik}\right) x'_j x'_k \\ &= \sum_{j=q+1}^{n} \sum_{k=q+1}^{n} \delta_{jk} x'_j x'_k = \sum_{i=q+1}^{n} (x'_i)^2. \end{aligned}$$

Furthermore, $\dfrac{\partial x_k}{\partial x'_a} = 0$ for $a = 1, \dots, q$, so that from the general formula that expresses coordinate vector fields from one coordinate system in terms of those from another, we have

$$\frac{\partial}{\partial x'_i} = \sum_{k=q+1}^{n} \frac{\partial x_k}{\partial x'_i} \frac{\partial}{\partial x_k} = \sum_{k=q+1}^{n} a_{ki} \frac{\partial}{\partial x_k}.$$

Hence if $(b_{ij}) = (a_{ij})^{-1}$, then

$$\begin{aligned} \sum_{i=q+1}^{n} x'_i \frac{\partial}{\partial x'_i} &= \sum_{i=q+1}^{n} \sum_{j=q+1}^{n} \sum_{k=q+1}^{n} b_{ij} x_j a_{ki} \frac{\partial}{\partial x_k} \\ &= \sum_{j=q+1}^{n} \sum_{k=q+1}^{n} \delta_{kj} x_j \frac{\partial}{\partial x_k} = \sum_{k=q+1}^{n} x_k \frac{\partial}{\partial x_k}. \end{aligned}$$ ∎

Geometrically N is the outward normal from every tubular hypersurface, and σ is the distance from the submanifold P, but these facts must be proved. That N is the outward normal requires the generalized Gauss Lemma (see Corollary 2.13) and must be postponed. But without further delay we can describe σ in terms of the distance function of M, and N in terms of velocities of geodesics.

Lemma 2.7. *Let $m \in M$, and suppose there is a unique unit-speed geodesic ξ from P to m meeting P orthogonally. Then*

$$\sigma(m) = \text{distance}(P, m) \tag{2.21}$$

and

$$N_{\xi(s)} = \xi'(s). \tag{2.22}$$

Therefore, σ is defined on $\exp_\nu(\mathcal{O}_P)$, and N is defined on $\exp_\nu(\mathcal{O}_P) - P$.

Proof. To obtain nice expressions for σ and N, we make a special choice of the Fermi coordinates $(x_1, \ldots, x_n)$. We are assuming that m is near enough to P so that there is a unique shortest geodesic ξ from m to P. Then we can assume that ξ has unit-speed and meets P orthogonally at $\xi(0) = p$ (cf. [Sakai, page 90]). Let b be such that $\xi(b) = m$. According to Lemma 2.5, there is a system of Fermi coordinates $(x_1, \ldots, x_n)$ centered at p such that $x_i\big(\xi(t)\big) = t\delta_{iq+1}$. Hence

$$\sigma(m) = x_{q+1}\big(\xi(b)\big) = b = \text{distance}(m, P).$$

Similarly,

$$N_{\xi(s)} = \frac{\partial}{\partial x_{q+1}}\bigg|_{\xi(s)} = \xi'(s). \qquad \square$$

We now derive the most important properties of N, σ and the Fermi fields. These properties will be used extensively and will allow us for the most part to avoid using Fermi coordinates themselves.

Lemma 2.8. *Let $X, Y \in \mathfrak{X}(P,p)^\perp$ and $A, B \in \mathfrak{X}(P,p)^\top$. Then on $\exp_\nu(\mathcal{O}_P)$ we have*

$$\begin{aligned}
\nabla_N N &= 0, && (2.23)\\
\|N\| &= 1, && (2.24)\\
N(\sigma) &= 1, && (2.25)\\
A(\sigma) &= 0, && (2.26)\\
[X,Y] = [A,B] &= [X,A] = [N,A] = 0, && (2.27)\\
[N,X] &= -\frac{1}{\sigma}X + \frac{1}{\sigma}X(\sigma)N, && (2.28)\\
[N,\sigma X] &= X(\sigma)N, && (2.29)\\
\nabla_N\nabla_N U + R_{NU}N &= 0, \quad \textit{for any } U \textit{ of the form } U = A + \sigma X. && (2.30)
\end{aligned}$$

Proof. Since (2.23)–(2.30) are local equations, we can use Fermi coordinates to establish them. Equations (2.23) and (2.24) are obvious from (2.22) and the definition of geodesic. For (2.25) we use the definitions of N and σ and compute

$$N(\sigma) = \frac{1}{2\sigma} N(\sigma^2) = \frac{1}{2\sigma} \sum_{i=q+1}^{n} \frac{x_i}{\sigma} \frac{\partial(\sigma^2)}{\partial x_i} = \frac{1}{\sigma^2} \sum_{i=q+1}^{n} x_i^2 = 1.$$

Since σ is independent of the tangential Fermi coordinates, we get (2.26). It is clear from the definition of Fermi field that (2.27) holds, because

$$\left[\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta} \right] = 0,$$

for $1 \leq \alpha, \beta \leq n$. To prove (2.28), we calculate as follows:

$$\begin{aligned} [N, X] &= - \sum_{i=q+1}^{n} X\left(\frac{x_i}{\sigma} \right) \frac{\partial}{\partial x_i} \\ &= \sum_{i=q+1}^{n} \left\{ -\frac{1}{\sigma} X(x_i) + \frac{x_i}{\sigma^2} X(\sigma) \right\} \frac{\partial}{\partial x_i} \\ &= -\frac{1}{\sigma} X + \frac{1}{\sigma} X(\sigma) N. \end{aligned}$$

Equation (2.29) follows easily from (2.28). To establish (2.30), we first compute

$$\begin{aligned} \nabla_N \nabla_N (\sigma X) &= \nabla_N [N, \sigma X] + \nabla_N \nabla_{\sigma X} N \qquad (2.31) \\ &= N X(\sigma) N - R_{N\,\sigma X} N \qquad \text{(by (2.29) and (2.23))} \\ &= [N, X](\sigma) N - R_{N\,\sigma X} N \quad \text{(by (2.25))} \\ &= \left\{ -\frac{1}{\sigma} X + \frac{1}{\sigma} X(\sigma) N \right\} (\sigma) N - R_{N\,\sigma X} N \\ &= -R_{N\,\sigma X} N. \end{aligned}$$

Also, (2.27) implies that

$$\nabla_N \nabla_N A = \nabla_N \nabla_A N = -R_{NA} N. \qquad (2.32)$$

Then (2.30) follows from (2.31) and (2.32). □

If ξ is a curve and X is a vector field along ξ, write $X' = \nabla_{\xi'} X$ and $X'' = \nabla_{\xi'} \nabla_{\xi'} X$.

Definition. *A vector field X along a geodesic ξ is called a* **Jacobi**[3] **field** *provided it satisfies the differential equation*

$$X'' + R_{\xi' X}\xi' = 0. \tag{2.33}$$

There is a close relation between Fermi fields and Jacobi fields:

Corollary 2.9. *Let ξ be a geodesic normal to P at p, and suppose that we have $X \in \mathfrak{X}(P,p)^{\perp}$ and $A \in \mathfrak{X}(P,p)^{\top}$. Then the restrictions to ξ*

$$\sigma X\Big|_{\xi} \qquad \text{and} \qquad A\Big|_{\xi}$$

are Jacobi fields.

Proof. This is an immediate consequence of (2.30). □

As a special case of Corollary 2.9 we see that Jacobi fields which vanish at a point $m \in M$ have a very simple description in terms of the normal coordinates at m.

Corollary 2.10. *Let $(x_1, \ldots, x_n)$ be a system of normal coordinates centered at $p \in P$, and let $m \in M$. Then along any radial geodesic the vector fields*

$$\sigma \frac{\partial}{\partial x_i}$$

are Jacobi fields for $1 \leq i \leq n$.

Jacobi fields have many uses in differential geometry, for example, in proving comparison theorems. For tubes, however, we prefer to use Fermi fields. In Section 3.1 we shall show that the principal curvature functions of the tubular hypersurfaces about a submanifold of any Riemannian manifold satisfy a Riccati equation that contains the same information as the Jacobi equation (2.33).

2.4 The Generalized Gauss Lemma

Let M be a Riemannian manifold and let $m \in M$. The classical[4] Gauss Lemma asserts that, although the exponential map $\exp_m$ is usually not an isometry, it is

[3]The Jacobi differential equation and its relation to Riemannian geometry was studied by Morse in his book [Morse]. The notion of Jacobi field is due to Ambrose [Am2].
[4]In paragraph 15 of his paper [Gauss2], Gauss states:

> Ductis in superficie curva ab eodem puncto initiali innumeris lineis brevissimis aequalis longitudinis, linea earum extremitates iungens ad illas singulas erit normalis,

which translates as

> If on a curved surface an infinite number of shortest lines [that is, geodesics] of equal length be drawn from the same initial point, the curve joining their extremities will be normal to each of the lines.

(The curve is a geodesic circle.) This is the first enunciation of what is now known as the **Gauss Lemma**. It is clear from his paper that Gauss was well aware of the importance of his result.

a partial isometry, as we now explain. Consider the tangent space M_m to be a Riemannian manifold isometric to $\mathbb{R}^n$. Call the geodesics in M_m emanating from $0 \in M_m$ **radial**, and call a tangent vector to M_m at $v \in M_m$ **radial** provided v is tangent to a radial geodesic. Similarly, call a tangent vector to M_m at v **spherical**, if it is perpendicular to the radial tangent vectors at v. The radial geodesics in M_m are mapped by $\exp_m$ into the geodesics in M starting at m. Now observe that the notions of *"radial"* and *"spherical"* also make sense for tangent vectors to M. Then the classical Gauss Lemma asserts two things: (1) $\exp_m$ preserves the lengths of radial tangent vectors, and (2) $\exp_m$ preserves the orthogonality between spherical and radial tangent vectors. On the other hand, most of the time $\exp_m$ does change the lengths and inner products of the spherical vectors; this accounts for the fact that it is almost never an isometry.

There is a generalization of the Gauss Lemma in which the point m is replaced by a submanifold P. However, the statement of this generalized Gauss Lemma looks different from the usual formulation of the ordinary Gauss Lemma. (But Lemma 2.11 below has been noted in the case that the submanifold is a point by Cheeger and Ebin [CE, page 9].) We first state and prove the generalized Gauss Lemma; then we show that it implies the Gauss Lemma as described above. Recall that for any $f \in \mathfrak{F}(M)$, the **gradient vector field** of f is defined to be the vector field $\operatorname{grad} f \in \mathfrak{X}(M)$ such that

$$\langle \operatorname{grad} f, X\rangle = Xf$$

for any $X \in \mathfrak{X}(M)$.

Lemma 2.11. (Generalized Gauss Lemma.) *Let P be a topologically embedded submanifold of a Riemannian manifold M. Then on $\exp_\nu(\mathcal{O}_P) - P$ we have $N = \operatorname{grad}\sigma$.*

Proof. Let $X \in \mathfrak{X}(P,p)^\perp$, $A \in \mathfrak{X}(P,p)^\top$, and write $U = A + \sigma X$. From (2.23), (2.29) and (2.30) it follows that

$$N^2\langle U, N\rangle = \left\langle \nabla_N \nabla_N U, N\right\rangle = -\left\langle R_{NU}N, N\right\rangle = 0. \tag{2.34}$$

Furthermore, using (2.23)–(2.29) we find that

$$\begin{aligned} N^2\langle U, \operatorname{grad}\sigma\rangle &= N^2(A + \sigma X)(\sigma) \\ &= N\big([N,A] + AN + [N,\sigma X] + (\sigma X)N\big)(\sigma) \\ &= N\big(X(\sigma)N\big)(\sigma) = 0 \end{aligned} \tag{2.35}$$

Thus from (2.34) and (2.35) we see that along any geodesic ξ normal to P the functions $t \longmapsto \langle U, N\rangle\big(\xi(t)\big)$ and $t \longmapsto \langle U, \operatorname{grad}\sigma\rangle\big(\xi(t)\big)$ are both linear functions

of t. To prove that they are the same linear function, it suffices to show that their limits and the limits of their first derivatives coincide at 0. In fact, we have

$$\langle U, N\rangle\big(\xi(0)\big) = \langle A + \sigma X, N\rangle\big(\xi(0)\big) = \langle A, \xi'(0)\rangle = 0,$$

and by (2.26) we have

$$\langle U, \operatorname{grad}\sigma\rangle\big(\xi(0)\big) = U(\sigma)\big(\xi(0)\big) = (A + \sigma X)(\sigma)\big(\xi(0)\big) = A(\sigma)\big(\xi(0)\big) = 0.$$

Furthermore, it follows from Lemma 2.8 that

$$\begin{aligned}
N\langle U, N\rangle\big(\xi(0)\big) &= \Big\langle \nabla_N(A + \sigma X), N\Big\rangle\big(\xi(0)\big) \\
&= \Big\{\Big\langle \nabla_A N, N\Big\rangle + \Big\langle N(\sigma)X + \sigma\nabla_N X, N\Big\rangle\Big\}\big(\xi(0)\big) \\
&= \Big\{\frac{1}{2}A\|N\|^2 + \langle X, N\rangle\Big\}\big(\xi(0)\big) \\
&= \langle X, N\rangle\big(\xi(0)\big),
\end{aligned}$$

and

$$\begin{aligned}
N\langle U, \operatorname{grad}\sigma\rangle\big(\xi(0)\big) &= N(A + \sigma X)(\sigma)\big(\xi(0)\big) \\
&= \Big\{[N, A] + X + \sigma N X\Big\}(\sigma)\big(\xi(0)\big) \\
&= X(\sigma)\big(\xi(0)\big).
\end{aligned}$$

But by Lemma 2.5 it is possible to choose the system of Fermi coordinates so that $x_\alpha\big(\xi(t)\big) = t\delta_{q+1\alpha}$ for $1 \le \alpha \le n$. Then

$$\begin{aligned}
\langle X, N\rangle\big(\xi(0)\big) &= \Big\langle \sum_{i=q+1}^{n} X(x_i)\frac{\partial}{\partial x_i}, \sum_{j=q+1}^{n} \frac{x_j}{\sigma}\frac{\partial}{\partial x_j}\Big\rangle\big(\xi(0)\big) \\
&= \Big\langle \sum_{i=q+1}^{n} X(x_i)\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_{q+1}}\Big\rangle\big(\xi(0)\big) \\
&= \Big\langle \sum_{i=q+1}^{n} X(x_i)\big(\xi(0)\big)e_i\,, e_{q+1}\Big\rangle \\
&= X(x_{q+1})\big(\xi(0)\big),
\end{aligned}$$

and

$$X(\sigma)=\frac{1}{2\sigma}X(\sigma^2)=\sum_{i=q+1}^{n}\frac{x_i}{\sigma}X(x_i),$$

so that $X(\sigma)\big(\xi(0)\big)=X(x_{q+1})\big(\xi(0)\big)$.

Thus the functions $t\longmapsto\langle U,N\rangle\big(\xi(t)\big)$ and $t\longmapsto\langle U,\operatorname{grad}\sigma\rangle\big(\xi(t)\big)$ are identically equal. Since ξ is arbitrary,

$$\langle U,N\rangle=\langle\operatorname{grad}\sigma,U\rangle. \tag{2.36}$$

For m near P every tangent vector in M_m is of the form U_m for some $U=A+\sigma X$ with $A\in\mathfrak{X}(P,p)^{\top}$ and $X\in\mathfrak{X}(P,p)^{\perp}$. Thus we conclude that $N=\operatorname{grad}\sigma$. □

Corollary 2.12. *The generalized Gauss Lemma implies the ordinary Gauss Lemma.*

Proof. Let $\{e_1,\dots,e_n\}$ be an orthonormal basis of M_m (or $(M_m)_0$). Then the normal coordinates on M_m associated with $\{e_1,\dots,e_n\}$ are just the dual basis of $\{e_1,\dots,e_n\}$. Let N_{M_m} be the normal for M_m and s the distance function.

By (2.25) we have $N(\sigma)=1$, and by the generalized Gauss Lemma we have $N=\operatorname{grad}\sigma$. As a result $\|N\|=1$. But the same argument also shows that $\|N_{M_m}\|=1$. Hence $\exp_m$ preserves the lengths of radial tangent vectors.

The usual interpretation of the gradient operator implies that $\operatorname{grad}\sigma$ is perpendicular to each of the hypersurfaces $\sigma=$ *constant*. A similar statement holds for M_m. In other words, the velocity vectors of radial geodesics are perpendicular to the hypersurfaces $\sigma=$ *constant* and $s=$ *constant*. Thus we see that $\exp_m$ also preserves the orthogonality between radial and spherical vectors. □

For the general case of a submanifold we have by the same reasoning:

Corollary 2.13. *The vector field N is the unit normal to each of the tubular hypersurfaces $\sigma=$ constant about a topologically embedded submanifold P of a Riemannian manifold M.*

The notions of radial vector and spherical vector for a submanifold can be defined much in the same way as for a point.

Definition. *Let P be a topologically embedded submanifold of a Riemannian manifold M, and let $p\in P$. We call a tangent vector $u\in P_p^{\perp}$ a* **radial vector** *and a tangent vector $x\in P_p$ a* **spherical vector**.

It is important to observe that the notions of radial vector and spherical vector make sense both for P considered as a submanifold of M, and P considered as the zero section of the normal bundle $\boldsymbol{\nu}$. We can now restate the generalized Gauss Lemma as:

Corollary 2.14. *Let P be a topologically embedded submanifold of a Riemannian manifold M. Then the exponential map $\exp_\nu\colon\boldsymbol{\nu}\longrightarrow M$ preserves the lengths of radial vectors and orthogonality between radial and spherical vectors.*

2.5 Problems

2.1 Let M be a Riemannian manifold and $m \in M$.

a. Show that the tangent space M_m is naturally a flat Riemannian manifold (that is, its curvature tensor vanishes identically).

b. For $x \in M_m$ let $\rho_x: \mathbb{R} \longrightarrow M_m$ be the curve defined by $\rho_x(t) = tx$. Show that ρ_x is a geodesic in M_m starting at m.

c. For $x \in M_m$ prove that the curve $t \longmapsto \xi_x(t)$ in M defined by $\xi_x = \exp_m \circ \rho_x$ is the unique geodesic in M that starts at m and has initial velocity x.

d. The manifold M_m is canonically identified with its tangent space $(M_m)_0$ at the origin as follows. For $x \in M_m$ let $x_0 \in (M_m)_0$ be the tangent vector defined by $x_0 = \rho_x'(0)$. Show that, in fact, the tangent map of the exponential map

$$\big((\exp_m)_*\big)_0 : (M_m)_0 \longrightarrow M_m$$

is this canonical identification.

e. Use the inverse function theorem to conclude that $\exp_m$ maps a neighborhood of $0 \in M_m$ diffeomorphically into M.

2.2 Let P be a topologically embedded submanifold of a Riemannian manifold M with normal bundle $\boldsymbol{\nu}$.

a. Prove that $\boldsymbol{\nu}$ is a differentiable manifold of the same dimension as M.

b. Let $\mathrm{Zero}(\boldsymbol{\nu}) = \{\, (p,0) \in \boldsymbol{\nu} \mid p \in P \,\}$. Show that $\mathrm{Zero}(\boldsymbol{\nu})$ is a submanifold of $\boldsymbol{\nu}$ and that $\exp_{\boldsymbol{\nu}}$ maps $\mathrm{Zero}(\boldsymbol{\nu})$ diffeomorphically onto P. For this reason we identify $\mathrm{Zero}(\boldsymbol{\nu})$ with P, and so we can regard P as a submanifold of $\boldsymbol{\nu}$.

c. Show that the normal bundle $\boldsymbol{\nu}$ is naturally a Riemannian manifold such that the projection $\pi: \boldsymbol{\nu} \longrightarrow M$ is a Riemannian submersion. (See [HK] or [BZ, page 242].)

2.3 Let M be an n-dimensional Riemannian manifold with tangent bundle $\pi: T(M) \longrightarrow M$.

a. Show that any system of local coordinates $(x_1, \dots, x_n)$ on M gives rise to a system of local coordinates $(x_1 \circ \pi, \dots, x_n \circ \pi, dx_1, \dots, dx_n)$ on $T(M)$. In particular, if $(x_1, \dots, x_n)$ is a system of normal coordinates on M, then $(x_1 \circ \pi, \dots, x_n \circ \pi, dx_1, \dots, dx_n)$ is a system of normal coordinates on $T(M)$.

b. Let P be a q-dimensional topologically embedded submanifold of a Riemannian manifold M. Show that if $(x_1, \dots, x_n)$ is a system of Fermi coordinates for P at $p \in P$, then $(x_1 \circ \pi, \dots, x_q \circ \pi, dx_{q+1}, \dots, dx_n)$ is a system of Fermi coordinates at $(p, 0) \in \boldsymbol{\nu}$ for the normal bundle $\boldsymbol{\nu}$, considered as a topologically embedded submanifold of $T(M)$.

2.4 Let P be a topologically embedded submanifold of $\mathbb{R}^n$. Show that the exponential map $\exp_\nu$ is given by $\exp_\nu(p, v) = p + v$ for $(p, v) \in \boldsymbol{\nu}$.

2.5 Let P be a topologically embedded submanifold of a sphere $S^n(\lambda)$ of radius $1/\sqrt{\lambda}$, regarded as sphere in $\mathbb{R}^{n+1}$ centered at the origin. Identify the normal bundle $\boldsymbol{\nu}$ of P in $S^n(\lambda)$ with $\{\, (p, v) \mid p \in P, v \in P_p^\perp \cap S^n(\lambda)_p \,\}$. With this identification show that for $(p, u) \in \boldsymbol{\nu}$ with $\|u\| = 1$ we have

$$\exp_\nu(p, tu) = p\cos(t\sqrt{\lambda}) + \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}u.$$

Chapter 3

The Riccati Equation for the Second Fundamental Forms

Our goal in this chapter is to study the geometry of a Riemannian manifold M in the neighborhood of a topologically embedded submanifold P. The principal tool that we shall use is the fact that the second fundamental forms of the tubular hypersurfaces about P satisfy a Riccati differential equation.

In Section 3.1 we define precisely the notions of tube and tubular hypersurface about a submanifold P of a Riemannian manifold M. Then we derive the Riccati differential equation for the second fundamental forms of the tubular hypersurfaces about P. The infinitesimal change of volume function of the exponential map $\exp_\nu$ is defined in Section 3.2; we show that it also satisfies a differential equation. In Section 3.3 we express the volume of a tube as an integral of the infinitesimal change of volume function.

The formulas in the first three sections hold for arbitrary Riemannian manifolds; there is no particular advantage to prove them only for submanifolds of Euclidean space. However, that is certainly an important special case, as is the special case when P is a point and M is a general Riemannian manifold. When the general formula for the volume of a tube is specialized in Theorem 3.15 to Euclidean space, the volume is expressed in terms of the second fundamental form of the submanifold, so it is not intrinsic. But it is used in Chapter 4 to derive Weyl's Tube Formula, which is intrinsic.

The Riccati differential equation for the second fundamental forms (see Corollary 3.3) and the differential equation for the infinitesimal change of volume function (see Theorem 3.11) provide information that is different from that of the more traditional Gauss and Codazzi equations. This information complements these two equations. Corollary 3.3 and Theorem 3.11 are useful in contexts other than Weyl's Tube Formula. For example, in Section 3.5 we derive some fundamental inequalities of Bishop and Günther for the volumes of geodesic balls. The nonintegrated

form of one of these inequalities is then used in Section 3.6 to give a simple proof of Myers' Theorem. In Chapter 8 we shall give a simultaneous generalization of Weyl's Tube Formula and the Bishop-Günther Inequalities using the Riccati differential equation for the second fundamental forms.

3.1 The Second Fundamental Forms of the Tubular Hypersurfaces

It is time we said precisely what we mean by a tube.

Definition. *Let P be a topologically embedded submanifold (possibly with boundary) in a Riemannian manifold M. Then a* **tube** $T(P,r)$ *of radius $r \geq 0$ about P is the set*

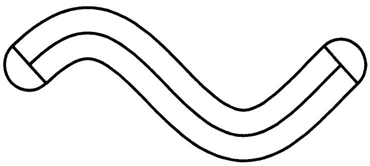
$$T(P,r) = \{\, m \in M \mid \text{ there exists a geodesic } \xi \text{ of length } L(\xi) \leq r \text{ from } m \text{ meeting } P \text{ orthogonally}\,\}. \tag{3.1}$$

Important Remarks.

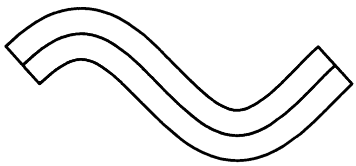
(1) If P has a boundary, then

$$T(P,r) \neq \{\, m \in M \mid \text{distance}(m,P) \leq r \,\}, \tag{3.2}$$

because the ends of the set on the right-hand side of (3.2) must be excluded to get the left-hand side.



Not a tube



A tube

(2) Recall that a Riemannian manifold is said to be **complete** provided that it is complete as a metric space. Although (3.1) makes sense in general, more often than not we shall assume that the ambient manifold M is complete.

(3) Unless otherwise stated, we shall not allow overlapping of a tube with itself.

(4) More stringently, until Chapter 8 we shall assume

$$\exp_\nu \colon \{\, (p,v) \in \boldsymbol{\nu} \mid \|v\| \leq r \,\} \longrightarrow T(P,r) \subset \exp_\nu(\mathcal{O}_P) \text{ is a diffeomorphism.} \tag{3.3}$$

If M is complete and the closure of P is compact, then condition (3.3) can always be achieved for sufficiently small $r > 0$.

(5) When (3.3) holds we can write (3.1) as

$$T(P,r) = \bigcup_{p \in P} \{\, \exp_p(v) \mid v \in P_p^{\perp} \text{ and } \|v\| \le r \,\}.$$

We shall also need a notion closely related to that of tube.

Definition. *We call a hypersurface of the form*

$$P_t = \{\, m \in T(P,r) \mid \operatorname{distance}(m,P) = t \,\}$$

the **tubular hypersurface** *at a distance t from P.*

For $0 < t \le r$ the tubular hypersurfaces P_t form a natural foliation of the tubular region $T(P,r) - P$. Let $S(t)$ be the second fundamental form of the hypersurface P_t (the precise definition of $S(t)$ is given below). We shall show that the function $t \longmapsto S(t)$ satisfies a Riccati differential equation by means of some simple calculations using Fermi fields.

Let ∇ and R denote the covariant derivative and curvature transformation of M. We use the notation R_N for the tensor field defined on the set $\exp_\nu(\mathcal{O}_P) - P$ by

$$R_N U = R_{NU} N,$$

for $U \in \mathfrak{X}(\exp_\nu(\mathcal{O}_P) - P)$. We shall also need another important operator.

Definition. *The* **shape operator**

$$S\colon \mathfrak{X}(\exp_\nu(\mathcal{O}_P) - P) \longrightarrow \mathfrak{X}(\exp_\nu(\mathcal{O}_P) - P)$$

is defined by

$$SU = -\nabla_U N. \tag{3.4}$$

Lemma 3.1. *S has the following properties*:

(i) *S is tensorial in the sense that $SfU = fSU$ for $U \in \mathfrak{X}(\exp_\nu(\mathcal{O}_P) - P)$ and $f \in \mathfrak{F}(\exp_\nu(\mathcal{O}_P) - P)$*;

(ii) *$\langle SU, V\rangle = \langle SV, U\rangle$ for $U, V \in \mathfrak{X}(\exp_\nu(\mathcal{O}_P) - P)$ with $\langle U, N\rangle = \langle V, N\rangle = 0$*;

(iii) *$SN = 0$.*

Proof. Part (i) is a consequence of the fact that the covariant derivative is linear with respect to functions in its first argument. Also, (iii) is obvious from (2.23). As for (ii), we use the generalized Gauss Lemma (Lemma 2.11), computing as follows:

$$\begin{aligned}
\langle SU, V\rangle - \langle SV, U\rangle &= -\langle \nabla_U N, V\rangle + \langle \nabla_V N, U\rangle \\
&= \langle [U,V], N\rangle \\
&= [U,V](\sigma) \\
&= U\langle V, N\rangle - V\langle U, N\rangle = 0.
\end{aligned}$$

□

The **covariant derivative of** S is defined by

$$\nabla_U(S)V = \nabla_U SV - S\nabla_U V$$

for $U, V \in \mathfrak{X}(\exp_\nu(\mathcal{O}_P) - P)$.

Lemma 3.2. *On* $\exp_\nu(\mathcal{O}_P) - P$ *we have*

$$\nabla_N(S) = S^2 + R_N. \tag{3.5}$$

Proof. Each tangent space M_m for $m \in \exp_\nu(\mathcal{O}_P) - P$ is spanned by vector fields of the form $U = A + \sigma X$, where $A \in \mathfrak{X}(P,p)^\top$ and $X \in \mathfrak{X}(P,p)^\perp$. Since (3.5) is an equation between tensor fields, it suffices to prove it by showing that both sides agree on vector fields of the form U. From (2.29) we have

$$[N, U] = X(\sigma)N. \tag{3.6}$$

Then using (2.23), (2.27), (3.6) and Lemma 3.1 we compute

$$\begin{aligned} \nabla_N(S)U &= \nabla_N(SU) - S\nabla_N U \\ &= -\nabla_N\nabla_U N - S[N,U] + S^2U \\ &= R_N U - \nabla_{[N,U]}N - S[N,U] + S^2U \\ &= R_N U + S^2U. \end{aligned} \tag{3.7}$$

□

For each t let $S(t)$ and $R(t)$ be the restrictions to the hypersurface P_t of S and R_N. Also, let $S'(t)$ be the restriction of $\nabla_N(S)$ to P_t. Then from Lemmas 3.1 and 3.2 we have:

Corollary 3.3. *On each tangent space to* P_t *the operators* $S(t)$ *and* $S'(t)$ *are symmetric. Furthermore,*

$$S'(t) = S(t)^2 + R(t). \tag{3.8}$$

Notice that (3.8) is a matrix Riccati differential equation; it contains the same information as the Jacobi differential equation (2.33). One is about as hard to solve as the other: the Riccati equation is first order but nonlinear, whereas the Jacobi equation is linear but of second order. We shall see shortly that each of the eigenvalues $\kappa_\alpha(t)$ of $S(t)$ satisfies a scalar Riccati differential equation. For the history of the Riccati differential equation and its relation to the Jacobi differential equation, see the introductions to the books [Reid1], [Reid2] and Chapter IV of [Watson].

We denote the **second fundamental form** of the submanifold P by T. Then T is more or less the limit of $S(t)$ as $t \longrightarrow 0$, but the limit must be taken along a specific normal geodesic to make sense. The precise definition of T is as follows.

Let A and B be vector fields on P that are tangent to P. X be a vector field on P that is normal to P. We define $T_A B$ to be the vector field normal to P such that

$$\langle T_A B, X\rangle = \langle \nabla_A B, X\rangle.$$

for all vector fields X normal to P. Then T is tensorial in the sense that it is linear with respect to functions $f \in \mathfrak{F}(P)$. Hence T gives rise to a multilinear function

$$T(p)\colon P_p \times P_p \times P_p^{\perp} \longrightarrow \mathbb{R}$$

for each $p \in P$. The value of $T(p)$ on $a, b \in P_p$ and $u \in P_p^{\perp}$ will be denoted by T_{abu}. Also, for $u \in P_p^{\perp}$ the **Weingarten map** $T_u\colon P_p \longrightarrow P_p$ is defined by

$$\langle T_u(a), b\rangle = T_{abu}.$$

It follows from the fact that $\nabla_A B - \nabla_B A = [A, B]$ is tangent to P that T_u is symmetric in a and b.

Of course, the tensor field S, when restricted to a tubular hypersurface, contains the same information as the second fundamental form of P_t, because $\langle T_A B, N\rangle = \langle SA, B\rangle$ for $A, B \in \mathfrak{X}(P,p)^{\top}$. More generally, for any hypersurface Q it is possible to define a tensor field S on Q by (3.4). Sometimes S is called the **shape operator** of the hypersurface Q (see [ON1, page 90], [ON4, page 107] and Chapter 10).

The shape operator is normally used to study the geometry of a particular hypersurface via the Gauss and Codazzi equations. Notice, however, that for tubes it is necessary to study a different aspect of S, namely how S varies from one tubular hypersurface to another. In fact, the fundamental equation describing this variation is (3.5).

Let ξ be a unit-speed geodesic normal to P at p with $\xi(0) = p$, and let $\{f_1, \dots, f_q\}$ be an orthonormal basis of P_p that diagonalizes the Weingarten map $T_{\xi'(0)}\colon P_p \longrightarrow P_p$. Extend these tangent vectors to unit vector fields $F_1(t), \dots, F_q(t)$ along ξ such that for each t and a, $F_a(t)$ is an eigenvector of $S(t)$. We write

$$S(t)F_a(t) = \kappa_a(t)F_a(t)$$

for $a = 1, \dots, q$. Let $\kappa_{q+2}(t), \dots, \kappa_n(t)$ be the remaining eigenvalues. For $i = q+2, \dots, n$ there are unit vector fields $F_{q+2}(t), \dots, F_n(t)$ along ξ such that

$$S(t)F_i(t) = \kappa_i(t)F_i(t).$$

Finally, we put $F_{q+1}(t) = \xi'(t)$; then $t \longmapsto \{F_1(t), \dots, F_n(t)\}$ is an orthonormal frame field along ξ.

Definition. *The functions* $\kappa_1, \dots, \kappa_q, \kappa_{q+2}, \dots, \kappa_n$ *are called the* **principal curvature functions** *of* S *and* $F_1, \dots, F_q, F_{q+2}, \dots, F_n$ *are called the* **principal curvature vector fields** *of* S.

More generally, the eigenfunctions of the shape operator of any orientable hypersurface are called the **principal curvatures** of the hypersurface. They depend up to sign on the choice of unit normal to the hypersurface. In the special case of an orientable piece of a tubular hypersurface P_t we can choose the outward normal. Then from the definitions we have

Lemma 3.4. *For fixed t the restrictions of the principal curvature functions to an orientable tubular hypersurface P_t are the principal curvatures of P_t.*

Although F_{q+1} is parallel along ξ, the other F_α's need not be. (However, frequently they can be chosen parallel when M is a symmetric space, for example a sphere or complex projective space. See Section 6.3.) Some technical difficulties arise at those values of t for which two of the κ_α's coincide. At such values of t one or more of the F_α's might be discontinuous and the κ_α's nondifferentiable. Nevertheless, there is still a Riccati differential equation.

Corollary 3.5. *Suppose F_α is differentiable at t. Then*

$$\kappa_\alpha'(t) = \kappa_\alpha(t)^2 + R_{\xi'(t)F_\alpha(t)\xi'(t)F_\alpha(t)}. \tag{3.9}$$

Proof. The differentiability of F_α and the equation $SF_\alpha = \kappa_\alpha F_\alpha$ imply the differentiability of κ_α. To establish (3.9), we use the fact that $\langle F_\alpha, F_\alpha' \rangle = 0$ and compute as follows:

$$\begin{aligned} R_{\xi' F_\alpha \xi' F_\alpha} &= \left\langle \nabla_{\xi'}(S)F_\alpha, F_\alpha \right\rangle - \|SF_\alpha\|^2 \\ &= \langle SF_\alpha, F_\alpha\rangle' - 2\langle SF_\alpha, F_\alpha' \rangle - \kappa_\alpha^2 \\ &= \kappa_\alpha' - \kappa_\alpha^2. \end{aligned}$$ □

Moreover, the following is true:

Corollary 3.6. *Let ρ^M denote the Ricci curvature of M. Then without exception*

$$\Bigg(\sum_{\substack{\alpha=1 \\ \alpha\neq q+1}}^{n} \kappa_\alpha \Bigg)' = \sum_{\substack{\alpha=1 \\ \alpha\neq q+1}}^{n} \kappa_\alpha^2 + \rho^M(N,N). \tag{3.10}$$

Proof. The function $\sum \kappa_a(t)$ is differentiable because it is just $\operatorname{tr} S(t)$. Hence Corollary 3.6 follows from Corollary 3.5. □

3.2 The Infinitesimal Change of Volume Function

In this section we derive a differential equation that will be used to describe how the exponential map $\exp_\nu : \boldsymbol{\nu} \longrightarrow M$ distorts volumes infinitesimally. It will be one of the fundamental tools for studying tube volumes.

Recall that a **Riemannian volume form** on an n-dimensional Riemannian manifold M is an n-form ω that assigns ± 1 to every orthonormal frame $\{e_1,\dots,e_n\}$. A Riemannian metric induces an inner product on the space of p-forms in a well-known fashion. With respect to this inner product it is easy to check that any Riemannian volume form ω has length 1. An orientable Riemannian manifold M has a globally defined Riemannian volume form ω; the choice of ω makes M an **oriented Riemannian manifold.**

Since we shall be studying the local geometry around a submanifold P of a Riemannian manifold M, we can assume without loss of generality that both manifolds are oriented. Let ω and ω_ν be the respective Riemannian volume forms of M and the normal bundle $\boldsymbol{\nu}$; then $\|\omega\| = \|\omega_\nu\| = 1$. The space of n-forms on each tangent space to either M or $\boldsymbol{\nu}$ has dimension 1, and so $\exp_\nu^*(\omega)$ must be a multiple of ω_ν. We make the choice of ω and ω_ν so that this multiple is positive. This leads to the following definition.

Definition. *Let P be a topologically embedded oriented submanifold of an oriented Riemannian manifold M. Then the function* chvol: $\exp_\nu(\mathcal{O}_P) \longrightarrow \mathbb{R}$ *is the everywhere nonnegative function given by*

$$\exp_\nu^*(\omega) = (\text{chvol} \circ \exp_\nu)\omega_\nu. \tag{3.11}$$

Lemma 3.7. *The restriction of* chvol *to P is identically* 1.

Proof. This is a consequence of Lemma 2.2. □

Definition. *Let $p \in P$ and let u be a unit vector in $P_p^\perp$. The* **infinitesimal change of volume function** *of P in the direction u is the function $t \longmapsto \vartheta_u(t)$, where*

$$\vartheta_u(t) = (\text{chvol} \circ \exp_\nu)(p, tu).$$

It is defined for $(p, tu) \in \mathcal{O}_P$.

The function ϑ_u measures the extent to which $\exp_\nu$ distorts volumes in the direction u. There is a description of $\vartheta_u(t)$ in terms of Fermi coordinates:

Lemma 3.8. *Let $(x_1, \dots, x_n)$ be a system of Fermi coordinates for P at $p \in P$ which has the same orientation as ω. (This just means that*

$$\omega\left(\frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_n}\right) > 0.)$$

Also, assume that the system of coordinates $(y_1, \dots, y_q)$ on P used to define the Fermi coordinates is chosen so that

$$\left\{ \frac{\partial}{\partial x_1}\bigg|_p, \dots, \frac{\partial}{\partial x_q}\bigg|_p \right\}$$

forms an orthonormal basis of P_p. *Then*

$$\vartheta_u(t) = \omega\left(\frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n}\right)(\exp_\nu(p, tu)). \tag{3.12}$$

Proof. It follows from Lemma 2.4 that all the tangent vectors

$$\left.\frac{\partial}{\partial x_1}\right|_p, \ldots, \left.\frac{\partial}{\partial x_n}\right|_p$$

are orthonormal. Then (3.12) follows from the definitions of Fermi coordinates and the function chvol when we evaluate both sides of (3.11) on

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}. \qquad \square$$

Lemma 3.9. *The infinitesimal change of volume function has the following properties:*

(i) *the tangent map of* $\exp_\nu$ *is singular along the geodesic* $t \longmapsto \exp_\nu(p, tu)$ *precisely where* $\vartheta_u(t)$ *vanishes for some* t;

(ii) *the orthonormal frame fields starting in normal directions used to define the system of oriented Fermi coordinates* $(x_1, \ldots, x_n)$ *can be rotated by any constant orientation preserving matrix without changing the definition of* $\vartheta_u(t)$;

(iii) $\vartheta_u(0) = 1$ *for all unit vectors* $u \in P_p^\perp$ *for all* $p \in P$.

Proof. It is easily seen that the infinitesimal volume function is just the Jacobian of the exponential map $\exp_\nu$. Then (i) follows from this fact or from (3.11). A proof of (ii) can be given along the lines of Lemma 2.6. Finally, (iii) follows from Lemma 3.7. $\square$

We shall need some elementary facts about the volume forms. Recall that a tensor field L of type $(1,1)$ on a Riemannian manifold M can be regarded as a function that assigns to each point $m \in M$ a linear transformation $L_m: M_m \longrightarrow M_m$.

Lemma 3.10. *Let* ω *be the volume form of an oriented Riemannian manifold* M *of dimension* n. *Then:*

(i) ω *is parallel, that is,* $\nabla_X(\omega) = 0$ *for all* $X \in \mathfrak{X}(M)$;

(ii) *for any tensor field* L *of type* $(1,1)$ *on* M *we have*

$$\sum_{i=1}^{n} \omega(X_1, \ldots, LX_i, \ldots, X_n) = \operatorname{tr}(L)\omega(X_1, \ldots, X_n) \tag{3.13}$$

for $X_1, \ldots, X_n \in \mathfrak{X}(M)$.

Proof. Since $\nabla_X \omega$ is an n-form, it is a multiple of ω. But ω has unit length, so that $0 = X\langle\omega,\omega\rangle = 2\langle\nabla_X\omega,\omega\rangle$; thus, $\nabla_X\omega = 0$, proving (i).

To prove (ii), we can suppose without loss of generality that $X_1, \ldots, X_n$ are linearly independent. We write $LX_i = \sum_{j=1}^n a_{ij}X_j$ and calculate as follows:

$$\begin{aligned}\sum_{i=1}^n \omega(X_1,\ldots,LX_i,\ldots,X_n) &= \sum_{ij=1}^n \omega(X_1,\ldots,a_{ij}X_j,\ldots,X_n)\\ &= \sum_{i=1}^n a_{ii}\omega(X_1,\ldots,X_i,\ldots,X_n)\\ &= \operatorname{tr}(L)\omega(X_1,\ldots,X_n).\end{aligned}$$

□

We are now ready to derive a fundamental differential equation that expresses the infinitesimal change of volume function in terms of the shape operator.

Theorem 3.11. *Let ξ be a unit-speed geodesic in $\exp_\nu(\mathcal{O}_P)$ normal to P with $\xi(0) = p \in P$ and $\xi'(0) = u$. Then along ξ for $t > 0$ and $(p, tu) \in \mathcal{O}_P$ we have*

$$\frac{\vartheta_u'(t)}{\vartheta_u(t)} = -\left(\frac{n-q-1}{t} + \operatorname{tr} S(t)\right). \tag{3.14}$$

Proof. By Lemma 2.5 we can choose a system of Fermi coordinates $(x_1, \ldots, x_n)$ so that

$$\left.\frac{\partial}{\partial x_{q+1}}\right|_{\xi(t)} = \xi'(t).$$

Write

$$\begin{aligned} A_a|_\xi &= \left.\frac{\partial}{\partial x_a}\right|_\xi, && (a = 1,\ldots,q),\\ N|_\xi &= \left.\frac{\partial}{\partial x_{q+1}}\right|_\xi, &&\\ X_i|_\xi &= \left.\frac{\partial}{\partial x_i}\right|_\xi, && (i = q+2,\ldots,n).\end{aligned}$$

Then we have

$$\vartheta_u(t) = \omega\big(A_1 \wedge \ldots \wedge A_q \wedge N \wedge X_{q+2} \wedge \ldots \wedge X_n\big)(\xi(t)). \tag{3.15}$$

From part (i) of Lemma 3.10 it follows that

$$N\omega\big(A_1 \wedge \ldots \wedge A_q \wedge N \wedge X_{q+2} \wedge \ldots \wedge X_n\big) \tag{3.16}$$

$$\begin{aligned}&= \sum_{a=1}^q \omega\big(A_1 \wedge \ldots \wedge \nabla_N A_a \wedge \ldots \wedge A_q \wedge N \wedge X_{q+2} \wedge \ldots \wedge X_n\big)\\ &\quad + \sum_{i=q+2}^n \omega\big(A_1 \wedge \ldots \wedge A_q \wedge N \wedge X_{q+2} \wedge \ldots \wedge \nabla_N X_i \wedge \ldots \wedge X_n\big).\end{aligned}$$

Since $[A_a, N] = 0$ and $[N, X_i] = \nabla_N X_i - \nabla_{X_i} N$, the right-hand side of (3.16) can be written as

$$\sum_{a=1}^{q} \omega\big(A_1 \wedge \ldots \wedge \nabla_{A_a} N \wedge \ldots \wedge A_q \wedge N \wedge X_{q+2} \wedge \ldots \wedge X_n\big) \tag{3.17}$$
$$+ \sum_{i=q+2}^{n} \omega\big(A_1 \wedge \ldots \wedge N \wedge X_{q+2} \wedge \ldots \wedge (\nabla_{X_i} N + [N, X_i]) \wedge \ldots \wedge X_n\big).$$

It follows from the definition of S and Lemma 2.8 that

$$\nabla_{A_a} N = -SA_a, \quad \nabla_{X_i} N = -SX_i \quad \text{and} \quad [N, X_i] = -\frac{1}{\sigma} X_i + \frac{1}{\sigma} X_i(\sigma) N,$$

and so (3.17) becomes

$$\sum_{a=1}^{q} \omega\big(A_1 \wedge \ldots \wedge (-SA_a) \wedge \ldots \wedge A_q \wedge N \wedge X_{q+2} \wedge \ldots \wedge X_n\big) \tag{3.18}$$
$$+ \sum_{i=q+2}^{n} \omega\big(A_1 \wedge \ldots \wedge N \wedge X_{q+2} \wedge \ldots \wedge (-SX_i - \frac{1}{\sigma} X_i) \wedge \ldots \wedge X_n\big).$$

Then from (3.13) and (3.16)–(3.18) it follows that

$$N\omega = -\left(\frac{n-q-1}{\sigma} + \operatorname{tr} S\right)\omega. \tag{3.19}$$

When both sides of (3.19) are evaluated on $\xi(t)$ and (3.15) is used, we get (3.14). Hence the theorem follows. □

3.3 The Volume of a Tube in Terms of the Infinitesimal Change of Volume Function

In this section we assume that P is a topologically embedded submanifold with compact closure of a complete Riemannian manifold M. For all $r \geq 0$ both $T(P, r)$ and P_r are measurable sets.

$$V_P^M(r) \;=\; \text{the } n\text{-dimensional volume of } T(P, r),$$

$$A_P^M(r) \;=\; \text{the } (n-1)\text{-dimensional volume of } P_r.$$

In Chapter 1 we gave an intuitive proof that $A_P^M(r)$ was the derivative of $V_P^M(r)$. In this section we prove this fact rigorously. First, we derive the formula for $A_P^M(r)$ in terms of the infinitesimal volume function. We shall make an assumption (namely (3.3)) in Lemma 3.12, Lemma 3.13 and Theorem 3.15 about the size of the tube $T(P, r)$. This restrictive assumption will be removed in Chapter 8.

Lemma 3.12. *Suppose that* $\exp_\nu: \{ (p,v) \in \boldsymbol{\nu} \mid \|v\| \le r \} \longrightarrow T(P,r)$ *is a diffeomorphism. Then*

$$\mathrm{A}_P^M(r) = r^{n-q-1} \int_P \int_{S^{n-q-1}(1)} \vartheta_u(r)\, du\, dP. \tag{3.20}$$

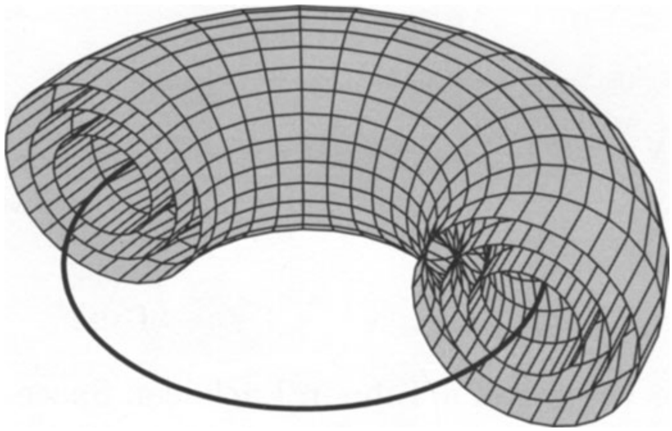
Proof. Let s be the function defined on a neighborhood of the zero section of the normal bundle $\boldsymbol{\nu}$ by

$$s(p,v) = \text{the Euclidean distance from } 0 \text{ to } v \text{ (that is, } s(p,v) = \|v\|)$$

for $(p,v) \in \boldsymbol{\nu}$. Then by the Generalized Gauss Lemma (Lemma 2.11) $s = \sigma \circ \exp_\nu$, where σ is the distance function given by (2.21). Let $*$ denote the Hodge star operator (either of M or $\boldsymbol{\nu}$). It is clear that each hypersurface

$$\{ (p,v) \mid v \in P_p^\perp,\ \|v\| = t,\ p \in P \}$$

in $\boldsymbol{\nu}$ has $*ds$ for its volume form, because $ds \wedge * ds$ is the volume form of $\boldsymbol{\nu}$. Moreover, $d\sigma \wedge * d\sigma$ is the volume form of M in a neighborhood of P. By the Generalized Gauss Lemma (Lemma 2.11) it follows that $*d\sigma$ is the volume form of P_t for each t.



Tubular hypersurfaces about a semicircle

Let $\exp_\nu^*$ denote the map induced by $\exp_\nu$ that maps differential forms on M to differential forms on $\boldsymbol{\nu}$. Since $\exp_\nu^*$ preserves wedge products, we have

$$\begin{aligned}\exp_\nu^*(d\sigma) \wedge \exp_\nu^*(*d\sigma) &= \exp_\nu^*(d\sigma \wedge * d\sigma) \\ &= \left\{ \omega\left(\frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n} \right) \circ \exp_\nu \right\} ds \wedge * ds.\end{aligned}$$

Furthermore, $\exp_\nu^*(d\sigma) = ds$, so that

$$\exp_\nu^*(*d\sigma) = \left\{ \omega\left(\frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n} \right) \circ \exp_\nu \right\} * ds. \tag{3.21}$$

Because P_r has compact closure, the integral of $*d\sigma$ over P_r is the volume $A_P^M(r)$. Therefore,

$$\mathrm{A}_P^M(r) = \int_{P_r} *d\sigma = \int_{\exp_\nu^{-1}(P_r)} \exp_\nu^*(*d\sigma) \tag{3.22}$$

$$= \int_P \int_{S^{n-q-1}(r)} \left\{ \omega\left(\frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n}\right) \circ \exp_\nu \right\} du\, dP,$$

where dP is the volume form of P and du is the volume form of $S^{n-q-1}(r)$. Now we use the map

$$h\colon S^{n-q-1}(1) \longrightarrow S^{n-q-1}(r)$$

given by $h(x) = rx$. Then $h^*(*ds) = r^{n-q-1}*ds$. Changing variables once more in (3.22) we get (3.20). ☐

Lemma 3.13. *Suppose that* $\exp_\nu\colon \{\,(p,v) \in \boldsymbol{\nu} \mid \|v\| \le r\,\} \longrightarrow T(P,r)$ *is a diffeomorphism. Then*

$$\frac{d}{dr}\mathrm{V}_P^M(r) = \mathrm{A}_P^M(r) = r^{n-q-1}\int_P \int_{S^{n-q-1}(1)} \vartheta_u(r)\, du\, dP. \tag{3.23}$$

Proof. Using the notation of Lemma 3.12, we compute as follows:

$$\begin{aligned}
\mathrm{V}_P^M(r) &= \int_{T(P,r)} d\sigma \wedge *d\sigma \\
&= \int_{\exp_\nu^{-1}(T(P,r))} \exp_\nu^*(d\sigma) \wedge \exp_\nu^*(*d\sigma) \\
&= \int_0^r \int_{\exp_\nu^{-1}(P_r)} ds \wedge \exp_\nu^*(*d\sigma) = \int_0^r \mathrm{A}_P^M(s)\, ds.
\end{aligned}$$

☐

3.4 The Volume of a Tube in Euclidean Space $\mathbb{R}^n$ in Terms of Its Second Fundamental Forms

So far we have derived three fundamental equations for tubes:

$$\boxed{\begin{aligned}
S'(t) &= S(t)^2 + R(t) \\
\frac{\vartheta_u'(t)}{\vartheta_u(t)} &= -\left(\frac{n-q-1}{t} + \operatorname{tr} S(t)\right) \\
\mathrm{V}_P^M(r) &= \int_0^r \int_P \int_{S^{n-q-1}(1)} t^{n-q-1}\vartheta_u(t)\, du\, dP\, dt
\end{aligned}}$$

These equations hold for any submanifold of any Riemannian manifold M when (3.3) holds. We now specialize them to a submanifold of $\mathbb{R}^n$.

Lemma 3.14. *Let P be a topologically embedded submanifold of $\mathbb{R}^n$. Then*

$$\vartheta_u(t) = \det(\delta_{ab} - t\,T_{abu}) \tag{3.24}$$

as long as the right-hand side of (3.24) *is nonnegative.*

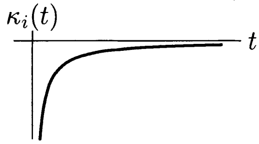
Proof. We compute $S(t)$ and afterwards $\vartheta_u(t)$. First, assume that the principal curvature functions of S are all differentiable. Then each satisfies the Riccati differential equation

$$\kappa_\alpha' = \kappa_\alpha^2; \tag{3.25}$$

in particular, each principal curvature function is a nondecreasing function. Thus there are two cases to distinguish: $\kappa_\alpha(0) = -\infty$ and $\kappa_\alpha(0)$ finite.

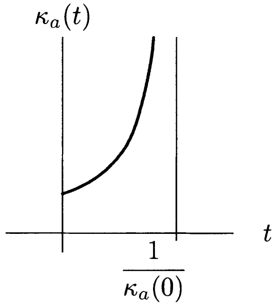
Case I: $q+2 \le i \le n$. Then $\kappa_i(0) = -\infty$, and so when we solve (3.25) we get

$$\kappa_i(t) = -\frac{1}{t}$$



Case II: $1 \le a \le q$. Then $\kappa_a(0)$ is finite. This time when we solve (3.25) we obtain

$$\kappa_a(t) = \frac{\kappa_a(0)}{1 - t\kappa_a(0)}.$$



The graph of $\kappa_a(t)$ when $\kappa_a(0) > 0$

Therefore, it follows from Theorem 3.11 that

$$\frac{\vartheta_u'(t)}{\vartheta_u(t)} = -\left(\operatorname{tr} S(t) + \frac{n-q-1}{t}\right) = \sum_{a=1}^{q} \frac{-\kappa_a(0)}{1 - t\kappa_a(0)}. \tag{3.26}$$

We integrate (3.26) and exponentiate; the result is

$$\vartheta_u(t) = \prod_{a=1}^{q}(1 - t\kappa_a(0))\vartheta_u(0) = \det(\delta_{ab} - t\,T_{abu})\vartheta_u(0). \tag{3.27}$$

In fact, by continuity (3.27) holds even at those points (if they exist) where a principal curvature function fails to be differentiable. On the other hand, by part (iii) of Lemma 3.9 we have $\vartheta_u(0) = 1$. Hence we obtain (3.24). □

Now we can derive a nonintrinsic formula for the volume of a tube about a submanifold of $\mathbb{R}^n$.

Theorem 3.15. *Let P be a topologically embedded submanifold of $\mathbb{R}^n$ with compact closure. Suppose that $\exp_\nu\colon \{ (p,v) \in \boldsymbol{\nu} \mid \|v\| \le r \} \longrightarrow T(P,r)$ is a diffeomorphism. Then*

$$V_P^{\mathbb{R}^n}(r) = \int_0^r \int_P \int_{S^{n-q-1}(1)} t^{n-q-1} \det(\delta_{ab} - t\,T_{abu})\, du\, dP\, dt. \qquad (3.28)$$

Proof. This is an immediate consequence of Lemmas 3.12, 3.13 and 3.14. □

3.5 The Bishop-Günther Inequalities

The three fundamental equations can also be used to study tubular regions in spaces of nonnegative or nonpositive curvature, as we shall see in detail in Chapter 8. In this section we start out more slowly by using the three fundamental equations to derive some inequalities due to Bishop [Bishop] and Günther [Gü], and then the theorem of Myers [Myers1].

It will be convenient to introduce some conventions for formulas involving trigonometric functions in order to avoid repetitive arguments. First, note that even when λ is negative, expressions such as $\sin(t\sqrt{\lambda})$ make sense and can be written in terms of hyperbolic functions. (Thus when $\lambda < 0$ we have

$$\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}} = \frac{\sinh(t\sqrt{|\lambda|})}{\sqrt{|\lambda|}}.) \qquad (3.29)$$

Similarly, when $\lambda = 0$ we interpret $\sin(t\sqrt{\lambda})/\sqrt{\lambda}$ as t.

First, we need an elementary lemma from calculus.

Lemma 3.16. *Let λ be a constant. Suppose that on the interval $(0,t_0)$ the function $k(t)$ satisfies the differential inequality and initial condition*

$$k' \ge k^2 + \lambda, \qquad k(0) = -\infty. \qquad (3.30)$$

Then

$$k(t) \ge \frac{-\sqrt{\lambda}}{\tan(t\sqrt{\lambda})} \qquad (3.31)$$

for $0 < t \le t_0$.

If the inequality in (3.30) is reversed, then the inequality in (3.31) is also reversed.

Proof. We do only the case $\lambda > 0$. Then we can rewrite (3.30) as

$$\frac{\dfrac{k'}{\sqrt{\lambda}}}{1+\dfrac{k^2}{\lambda}} \geq \sqrt{\lambda}, \tag{3.32}$$

Integrating both sides of (3.32) and using the assumption that $k(0) = -\infty$, we obtain

$$\arctan\left(\frac{k(t)}{\sqrt{\lambda}}\right) \geq t\sqrt{\lambda} - \frac{\pi}{2},$$

which is equivalent to (3.31). $\square$

Let M be a Riemannian manifold and let $m \in M$. A point $m' \in M$ is said to be **conjugate** to m if the tangent map of the exponential map $\exp_m$ is singular somewhere on the set $\exp_m^{-1}(m')$. Next let ξ be a geodesic starting at m. The first point on ξ where ξ ceases to minimize distance is called the **cut point** of m **along the geodesic** ξ. The **cut locus** of m is the set of cut points of m.

It is known that the cut point along a geodesic always occurs before any conjugate points (see, for example, [ON4, page 271] and [Ko]). More precisely, if $S(M) = \{\, u \in M_m \mid m \in M, \ \|u\| = 1 \,\}$ denotes the **unit sphere bundle** of M, we can define functions

$$e_{\text{cut}}, e_{\text{conj}} \colon S(M) \longrightarrow \mathbb{R}$$

by

$$\begin{aligned} e_{\text{cut}}(m,u) &= \sup\{\, t > 0 \mid \text{distance}(\exp_m(tu), m) = t \,\}, \\ e_{\text{conj}}(m,u) &= \inf\{\, t > 0 \mid \text{kernel}((\exp_m)_{*(tu)}) \neq 0 \,\}. \end{aligned}$$

Then $e_{\text{conj}}(m,u)$, if finite, is the distance along the geodesic $t \longmapsto \exp_m(p,tu)$ to its first conjugate point. As noted above $e_{\text{cut}}(m,u) \leq e_{\text{conj}}(m,u)$ for all $u \in M_m$. See Chapter 8 for more information and a generalization of these notions to tubes.

A geodesic ball about a point is a particularly simple case of a tube. Notice that for a geodesic ball the focal assumption (3.3) can be rephrased in more familiar terms by saying that at most the boundary of the geodesic ball about a point m can intersect the cut locus of m. Also, (2.2) reduces to

$$\mathcal{O}_m = \{\, tu \in M_m \mid \|u\| = 1, \ 0 \leq t < e_{\text{cut}}(m,u)\}.$$

Now we are ready to prove a fundamental relation between curvature and volume.

Theorem 3.17. (The Bishop-Günther Inequalities.) *Let M be a complete Riemannian manifold and assume that r is not greater than the distance between m and its cut locus. (So any point $m' \in M$ with* distance$(m,m') \leq r$ *has a unique unit-speed*

geodesic connecting with m.) Let K^M denote the sectional curvature of M and let λ be a constant. Then

$$K^M \geq \lambda \quad \text{implies} \quad V_m^M(r) \leq \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \int_0^r \left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-1} dt; \tag{3.33}$$

$$K^M \leq \lambda \quad \text{implies} \quad V_m^M(r) \geq \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \int_0^r \left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-1} dt. \tag{3.34}$$

Proof. For each $i = 1, \ldots, n$ we apply Lemma 3.16 to the function $k(t) = \kappa_i(t)$. Each principal curvature function satisfies the Riccati differential inequality

$$\kappa_i' \geq \kappa_i^2 + \lambda,$$

because the curvature of M satisfies

$$R_{\xi'(t)F_i(t)\xi'(t)F_i(t)} \geq \lambda.$$

Therefore, by Lemma 3.16 we obtain

$$\kappa_i(t) \geq \frac{-\sqrt{\lambda}}{\tan(t\sqrt{\lambda})}.$$

Summing the principal curvature functions, we get

$$\operatorname{tr} S(t) \geq \frac{-(n-1)\sqrt{\lambda}}{\tan(t\sqrt{\lambda})}, \tag{3.35}$$

and so by Theorem 3.11

$$\frac{\vartheta_u'(t)}{\vartheta_u(t)} \leq -\frac{n-1}{t} + \frac{(n-1)\sqrt{\lambda}\cos(t\sqrt{\lambda})}{\sin(t\sqrt{\lambda})}. \tag{3.36}$$

This inequality is easily integrated. Using the fact that $\vartheta_u(0) = 1$, it follows from (3.36) that

$$\vartheta_u(t) \leq \left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{n-1}.$$

Now Theorem 3.17 follows from this equation and Lemma 3.13. $\square$

Günther [Gü] proved (3.34), and Bishop [Bishop], [BC, page 256] proved (3.33), as well as a sharper result (Theorem 3.19 below) making use of the Ricci curvature instead of the sectional curvature. We shall show in Section 8.3 of Chapter 8 that there is a simultaneous generalization of Weyl's Tube Formula and the Bishop-Günther Inequalities.

A Riemannian manifold $\mathbb{K}^n(\lambda)$ is called a **space of constant curvature** provided the sectional curvature of $\mathbb{K}^n(\lambda)$ is identically equal to a constant λ. Any space of constant curvature is locally isometric to one of the following spaces:

Euclidean space	$\mathbb{R}^n$	$(\lambda = 0)$,
a sphere	$S^n(\lambda)$	$(\lambda > 0)$,
hyperbolic space	$H^n(\lambda)$	$(\lambda < 0)$.

As a special case of Theorem 3.17 we obtain:

Corollary 3.18. *The volume $V_m^{\mathbb{K}^n(\lambda)}(r)$ of a geodesic ball in a complete space $\mathbb{K}^n(\lambda)$ of constant sectional curvature λ is given by*

$$V_m^{\mathbb{K}^n(\lambda)}(r) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \int_0^r \left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-1} dt, \tag{3.37}$$

where r is not greater than the distance between m and its cut locus.

Thus for a sphere $S^n(\lambda)$ equation (3.37) holds for $r \leq \pi/\sqrt{\lambda}$. Of course, in (3.37) the integration can be carried out, but only in such a messy fashion that it is probably better to leave formula (3.37) as is. However, see Section A.A.3 of the Appendix for more information about the right-hand side of (3.37), including a graph.

Bishop's improvement [Bishop] of Theorem 3.17 is as follows.

Theorem 3.19. (Bishop's Theorem.) *Let M be a complete Riemannian manifold and assume that r is not greater than the distance between m and its cut locus. Let λ be a nonnegative constant and assume that the Ricci curvature of M satisfies*

$$\rho^M(x,x) \;\geq\; (n-1)\lambda\|x\|^2$$

for all tangent vectors x to M. Then for all unit tangent vectors u to M and all $t \leq r$ we have

$$\vartheta_u(t) \leq \left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{n-1}. \tag{3.38}$$

Furthermore,

$$V_m^M(r) \leq \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \int_0^r \left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-1} dt. \tag{3.39}$$

Proof. Let

$$f(t) = \frac{1}{n-1} \operatorname{tr} S(t) = \frac{1}{n-1} \sum_{\alpha=1}^{n-1} \kappa_\alpha(t).$$

It follows from Corollary 3.6 that

$$f' \geq \frac{1}{n-1} \sum_{\alpha=1}^{n-1} \kappa_\alpha^2 + \lambda.$$

By the Cauchy-Schwarz Inequality we have

$$\begin{aligned}(n-1)^2 f^2 &= \left(\sum_{\alpha=1}^{n-1} \kappa_\alpha \cdot 1 \right)^2 \\ &\leq \left(\sum_{\alpha=1}^{n-1} \kappa_\alpha^2 \right) \left(\sum_{\alpha=1}^{n-1} 1 \right) = (n-1) \left(\sum_{\alpha=1}^{n-1} \kappa_\alpha^2 \right),\end{aligned}$$

and so

$$f'(t) \geq f(t)^2 + \lambda.$$

We apply Lemma 3.16 using the function $f(t)$ for $k(t)$. Thus we again obtain (3.35). Then the rest of the proof of Theorem 3.19 is the same as that of Theorem 3.17. □

3.6 Myers' Theorem

Recall that a Riemannian metric gives rise on a manifold M to a distance function $(p, q) \longmapsto \operatorname{distance}(p, q)$ in the sense of metric spaces. The **diameter** of a metric space M (in particular, a Riemannian manifold) is defined as

$$\operatorname{diameter}(M) = \sup_{p,q \in M} \operatorname{distance}(p, q).$$

The following lemma is an immediate consequence of the definitions.

Lemma 3.20. *If M is a complete Riemannian manifold, then*

$$\operatorname{diameter}(M) = \sup_{(m,u) \in S(M)} e_{\text{cut}}(m, u).$$

We can now give a proof of one of the most important theorems in global Riemannian geometry, Myers' Theorem [Myers1]. We derive it as a consequence of (3.38).

Theorem 3.21. (Myers'[1] Theorem.) *Let M be a complete Riemannian manifold and suppose that there is a number $\lambda > 0$ such that*

$$\rho^M(x,x) \geq (n-1)\lambda \|x\|^2 \tag{3.40}$$

for all tangent vectors x to M. Then M is compact, the fundamental group of M is finite, and the diameter of M satisfies

$$\text{diameter}(M) \leq \frac{\pi}{\sqrt{\lambda}}. \tag{3.41}$$

Proof. Let $t \longmapsto \vartheta_u(t)$ be the infinitesimal change of volume function with respect to any point $m \in M$, where u is any unit tangent vector in M_m. Equation (3.40) implies that (3.38) holds, and from it we see that $\vartheta_u(t)$ must vanish somewhere in the interval $(0, \pi/\sqrt{\lambda})$. We know that the vanishing of $\vartheta_u(t)$ is equivalent to the singularity of the tangent map of the exponential map $\exp_m$. Hence there are conjugate points along every geodesic emanating from every point $m \in M$, and, in fact, from (3.38) we have

$$e_{\text{cut}}(m,u) \leq e_{\text{conj}}(m,u) \leq \frac{\pi}{\sqrt{\lambda}} \tag{3.42}$$

for all $u \in M_m$. Therefore, (3.41) follows from Lemma 3.20 and (3.42). Since M is a complete metric space with finite diameter, it is compact.

The universal covering space $\tilde{M}$ of M is also complete and the Ricci curvature of $\tilde{M}$ satisfies (3.40), so $\tilde{M}$ is compact by exactly the same argument. The number of elements in the fundamental group of M equals the number of points in any fiber of the covering $\tilde{M} \longrightarrow M$. Since both M and $\tilde{M}$ are compact, this number must be finite. □

Bishop [Bishop] observed that it is possible to modify the proof of Myers' Theorem to estimate the volume of a Riemannian manifold with Ricci curvature bounded below by λ. We now prove this estimate.

Theorem 3.22. *Let M be a compact n-dimensional manifold and $S^n(\lambda)$ an n-dimensional sphere of constant curvature λ. Suppose that*

$$\rho^M(x,x) \geq (n-1)\lambda \|x\|^2 \tag{3.43}$$

for all tangent vectors x to M. Then

$$\text{volume}(M) \leq \text{volume}(S^n(\lambda)) = \frac{2\pi^{\frac{1}{2}(n+1)}}{\Gamma(\frac{1}{2}(n+1))\lambda^{\frac{1}{2}n}}. \tag{3.44}$$

[1] Sumner B. Myers (1910–1955). American Mathematician. Myers graduated *summa cum laude* from Harvard in 1929, then wrote his dissertation under the direction of Marston Morse,. After several fellowships, he came to the University of Michigan in 1936. Although Myers published only 21 papers, he has had a great influence in differential geometry. See [Myers2].

Proof. By Myers' Theorem the universal covering space $\tilde{M}$ of M is compact; moreover, volume$(\tilde{M}) = p$ volume(M), where p is the number of elements in any fiber of $\tilde{M} \longrightarrow M$. Thus the volume of M is not larger than the volume of $\tilde{M}$. Therefore, without loss of generality we can assume that M is simply connected.

Since the cut locus of any point $m \in M$ has measure zero, we have

$$\mathrm{volume}(M) = \mathrm{volume}(\exp_m(\mathcal{O}_m)). \tag{3.45}$$

We use $\exp_m$ to transfer the integration from M to M_m:

$$\begin{aligned} \mathrm{volume}(\exp_m(\mathcal{O}_m)) &= \int_{\exp_m(\mathcal{O}_m)} \omega = \int_{\mathcal{O}_m} \exp_m^*(\omega) \\ &= \int_{\mathcal{O}_m} (\mathrm{chvol} \circ \exp_m)\, \omega_{M_m}. \end{aligned} \tag{3.46}$$

Next we change to polar coordinates in M_m. The distance from m to any point $m' \in \exp_m(\mathcal{O}_m)$ is less than or equal to $e_{\mathrm{cut}}(m,u)$, where $t \longmapsto \exp_m(tu)$ is the geodesic from m passing through m'. Hence

$$\int_{\mathcal{O}_m} (\mathrm{chvol} \circ \exp_m)\, \omega_{M_m} \le \int_{S^{n-1}(1)} \int_0^{e_{\mathrm{cut}}(m,u)} t^{n-1} \vartheta_u(t)\, dt\, du. \tag{3.47}$$

From (3.42) and (3.38) it follows that

$$\begin{aligned} \int_{S^{n-1}(1)} \int_0^{e_{\mathrm{cut}}(m,u)} t^{n-1} \vartheta_u(t)\, dt\, du &\le \int_{S^{n-1}(1)} \int_0^{\pi/\sqrt{\lambda}} \left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}} \right)^{n-1} dt \\ &= \frac{2}{\Gamma(\frac{n}{2})} \left(\frac{\pi}{\lambda}\right)^{n/2} \int_0^{\pi} (\sin s)^{n-1}\, ds. \end{aligned} \tag{3.48}$$

Combining (3.45)–(3.48), we obtain

$$\mathrm{volume}(M) \le \frac{2}{\Gamma(\frac{n}{2})} \left(\frac{\pi}{\lambda}\right)^{n/2} \int_0^{\pi} (\sin s)^{n-1}\, ds. \tag{3.49}$$

For $M = S^n(\lambda)$ the inequalities (3.47)–(3.49) become equalities, and so

$$\mathrm{volume}(S^n(\lambda)) = \frac{2}{\Gamma(\frac{n}{2})} \left(\frac{\pi}{\lambda}\right)^{n/2} \int_0^{\pi} (\sin s)^{n-1}\, ds. \tag{3.50}$$

Then (3.44) follows from (3.49) and (3.50). □

3.7 Problems

3.1 Recall from Section 3.4 that a Riemannian manifold $\mathbb{K}^n(\lambda)$ is said to have **constant curvature** provided there exists a constant λ such that

$$K(\Pi_p) = \lambda$$

for any 2-dimensional subspace Π_p to $\mathbb{K}^n(\lambda)$. Show that the curvature tensor field of a space of constant curvature is given by the formula

$$R_{WXYZ} = \lambda\Big\{\langle W, Y\rangle\langle X, Z\rangle - \langle W, Z\rangle\langle X, Y\rangle\Big\}$$

for $W, X, Y, Z \in \mathfrak{X}(\mathbb{K}^n(\lambda))$. (This formula can be written more concisely as

$$R_{WXYZ} = \lambda\langle W \wedge X, Y \wedge Z\rangle.)$$

3.2 Show that the Ricci and scalar curvatures of a space $\mathbb{K}^n(\lambda)$ of constant curvature λ are given by

$$\rho(X, Y) = (n-1)\lambda\langle X, Y\rangle \quad \text{and} \quad \tau = n(n-1)\lambda$$

for $X, Y \in \mathfrak{X}(\mathbb{K}^n(\lambda))$.

3.3 Let P be a submanifold of a space $\mathbb{K}^n(\lambda)$ of constant curvature λ, and let S be the shape operator of the tubular hypersurfaces of P. Show that each of the principal curvature functions κ_α of S satisfies the differential equation

$$\kappa'_\alpha = \kappa_\alpha^2 + \lambda. \tag{3.51}$$

3.4 Equation (3.51) has already been solved in Theorem 3.17 for the case when $\kappa_\alpha(0) = -\infty$. Solve it also for the case when $\kappa_\alpha(0)$ is finite.

3.5 Show that for any topologically embedded submanifold P of a space $\mathbb{K}^n(\lambda)$ of constant curvature λ the infinitesimal change of volume function for P is given by the formula

$$\vartheta_u(t) = (\cos(t\sqrt{\lambda}))^q \left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}\,t}\right)^{n-q-1} \det\left(\delta_{ab} - \left(\frac{\tan(t\sqrt{\lambda})}{\sqrt{\lambda}}\right) T_{abu}\right).$$

Conclude that the volume of a tube of radius r about a submanifold P in a space $\mathbb{K}^n(\lambda)$ of constant curvature λ is given by

$$\frac{d}{dr}V_P^{\mathbb{K}^n(\lambda)}(r) = A_P^{\mathbb{K}^n(\lambda)}(r) = \big(\cos(r\sqrt{\lambda})\big)^q \left(\frac{\sin(r\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-q-1} \tag{3.52}$$

$$\cdot \int_P \int_{S^{n-q-1}(1)} \det\left(\delta_{ab} - \left(\frac{\tan(r\sqrt{\lambda})}{\sqrt{\lambda}}\right) T_{abu}\right) du\, dP$$

for r not too large.

3.6 Try to integrate (3.52) from 0 to r and see what complications arise.

3.7 Let $t \longmapsto U(t)$ be a curve in the manifold of $n \times n$ matrices and let $t \longmapsto S(t)$ and $t \longmapsto R(t)$ be curves in the manifold of symmetric $n \times n$ matrices. Consider the differential equations

$$S' = S^2 + R, \tag{3.53}$$

$$U' + SU = 0, \tag{3.54}$$

$$U'' + RU = 0. \tag{3.55}$$

What are the relations among (3.53), (3.54) and (3.55)?

3.8 Prove the **Cartan-Hadamard Theorem**: *Let M be a connected complete Riemannian manifold with nonpositive sectional curvature. Then for any $m \in M$ the exponential map $\exp_m: M_m \longrightarrow M$ is a covering map. In particular, if M is simply connected it must be diffeomorphic to $\mathbb{R}^n$.*

3.9 What goes wrong if one tries to generalize the Cartan-Hadamard Theorem by using the exponential map $\exp_\nu$ of the normal bundle $\boldsymbol{\nu}$ of a submanifold instead of the exponential map of a point?

3.10 Prove that the following conditions are equivalent:

(i) p and q are conjugate.

(ii) There is a nontrivial Jacobi field along the geodesic from p to q that vanishes at both p and q.

(iii) At least one of the principal curvature functions of the geodesic balls centered at p blows up as the radii of the balls approach distance(p, q).

3.11 What goes wrong when one tries to sharpen the Cartan-Hadamard Theorem by assuming nonpositive Ricci curvature instead of nonpositive sectional curvature?

3.12 Show that for appropriate λ_1 and λ_2 the Riemannian manifold

$$S^n(\lambda_1) \times H^n(\lambda_2)$$

has positive scalar curvature. Conclude that it is impossible to replace "Ricci curvature" with "scalar curvature" in Myers' Theorem.

3.13 Let $f: M \longrightarrow \mathbb{R}^n$ be an integrable function. Show that the integral of f over the tubular hypersurface P_r is given by

$$\int_{P_r} f{*}d\sigma = r^{n-q-1} \int_P \int_{S^{n-q-1}(1)} f(\exp_\nu(r\,u))\vartheta_u(r)\,du\,dP. \tag{3.56}$$

Chapter 4

The Proof of Weyl's Tube Formula

In Theorem 3.15 we wrote down a formula for the volume $V_P^{\mathbb{R}^n}(r)$ of a tube about a submanifold P of Euclidean space $\mathbb{R}^n$. Although this formula has a great deal of interest, it is not our principal concern, because the integrand is a function of the second fundamental form T of P. As Weyl says in [Weyl1]:

> **So far we have hardly done more than what could have been accomplished by any student in a course of calculus.**

Thus in this chapter we shall be dealing with the deepest part of Weyl's paper, in which he reexpresses the tube volume $V_P^{\mathbb{R}^n}(r)$ entirely in terms of the curvature tensor of P. This means that tube volume is intrinsic to P, because in contrast to the second fundamental form, the curvature tensor does not depend on the particular way that P is embedded in $\mathbb{R}^n$.

To complete the proof of his tube formula, Weyl[1] appeals at the very end of the paper to the theory of invariants, which he had developed previously in [Weyl2]. We explain without proofs that part of the theory that we need in a modern setting. Actually, it is possible to avoid the general theory of invariants (see the second proof of Corollary 4.6 below, and also [Sadov]), but certainly the theory of invariants sheds a great deal of light on the tube formula.

We shall need a generalization of the curvature tensor of a Riemannian manifold. Therefore, in Section 4.1 we introduce the notion of double form; it is a formalism that is useful for writing down the coefficients in Weyl's Tube Formula

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[1] Hermann Klaus Hugo Weyl (1885–1955).
It is not possible in a mere footnote to summarize the vast output of Weyl. For details the excellent articles by C.N Yang, R. Penrose and A. Borel in [Chan] should be consulted, as well as the many other biographical notes. Suffice it to say that Penrose has called Weyl the greatest mathematician whose work lies entirely in the 20^{th} century.

in terms of curvature. Section 4.2 contains a brief discussion of that part of the theory of invariants that we need for the tube formula, and also for the power series expansions of Chapter 9. Then in Section 4.3 invariants are used to compute moments, that is, to express integrals of invariants over spheres as finite sums. However, for symmetric invariants these integrals can be computed (as in the second proof of Corollary 4.6) without appealing to the general theory of invariants. The proof of the tube formula is completed in Section 4.4. The chapter is concluded in Section 4.5 with a discussion of generalizations of the tube formula; this discussion will be continued in subsequent chapters.

4.1 Double Forms

Let M be any C^∞ differentiable manifold, and denote by $\mathfrak{F}(M)$ the algebra of differentiable functions on M. Following de Rham [dR, pages 30–33] we define a **double form of type** (p, q) on M to be an $\mathfrak{F}(M)$-linear map

$$\alpha\colon \mathfrak{X}(M)^p \times \mathfrak{X}(M)^q \longrightarrow \mathfrak{F}(M)$$

which is antisymmetric in the first p variables and also in the last q. We use the notation

$$\alpha(X_1, \dots, X_p)(Y_1, \dots, Y_q) \tag{4.1}$$

to denote the value of α on vector fields $X_1, \dots, X_p, Y_1, \dots, Y_q$. Then

$$\alpha(X_1, \dots, X_p)\colon \mathfrak{X}(M)^q \longrightarrow \mathfrak{F}(M)$$

is an $\mathfrak{F}(M)$-linear map whose value on vector fields $Y_1, \dots, Y_q$ is given by (4.1). Most important for us will be the **symmetric double forms**. These are the double forms for which $p = q$ and

$$\alpha(X_1, \dots, X_p)(Y_1, \dots, Y_p) = \alpha(Y_1, \dots, Y_p)(X_1, \dots, X_p).$$

for all $X_1, \dots, X_p, Y_1, \dots, Y_p \in \mathfrak{X}(M)$.

It is possible to define an **exterior product** $\alpha \wedge \beta$ of double forms α and β in much the same way as the exterior product between ordinary differential forms is defined. Let α have type (p, q) and β type (r, s). Let ε_ρ denote the sign of the permutation ρ. Then:

$$\begin{aligned}(\alpha \wedge \beta)&(X_1, \dots, X_{p+r})(Y_1, \dots, Y_{q+s}) \\ &= \sum \varepsilon_\rho\, \varepsilon_\sigma\, \alpha(X_{\rho_1}, \dots, X_{\rho_p})(Y_{\sigma_1}, \dots, Y_{\sigma_q}) \\ &\qquad \cdot\beta(X_{\rho_{p+1}}, \dots, X_{\rho_{p+r}})(Y_{\sigma_{q+1}}, \dots, Y_{\sigma_{q+s}}),\end{aligned} \tag{4.2}$$

where the sum is taken over all $\rho \in \mathrm{Sh}(p, r)$ and $\sigma \in \mathrm{Sh}(q, s)$. Here

$$\mathrm{Sh}(p, r) = \left\{\, \rho \in \mathfrak{S}_{p+r} \mid \rho_1 < \rho_2 < \cdots < \rho_p \text{ and } \rho_{p+1} < \cdots < \rho_{p+r} \,\right\},$$

where $\mathfrak{S}_{p+r}$ is the symmetric group of degree $p+r$ and "Sh" is an abbreviation for "shuffle". So an alternate version of (4.2) is

$$(\alpha \wedge \beta)(X_1, \ldots, X_{p+r})(Y_1, \ldots, Y_{q+s})$$

$$= \frac{1}{p!q!r!s!} \sum_{\substack{\rho \in \mathfrak{S}_{p+r} \\ \sigma \in \mathfrak{S}_{q+s}}} \varepsilon_\rho \, \varepsilon_\sigma \, \alpha(X_{\rho_1}, \ldots, X_{\rho_p})(Y_{\sigma_1}, \ldots, Y_{\sigma_q})$$

$$\cdot \beta(X_{\rho_{p+1}}, \ldots, X_{\rho_{p+r}})(Y_{\sigma_{q+1}}, \ldots, Y_{\sigma_{q+s}}).$$

It is straightforward to show that $\wedge$ is associative, and also commutative in the sense that

$$\alpha \wedge \beta = (-1)^{pr+qs} \beta \wedge \alpha.$$

The double forms of type $(p, 0)$ are just the ordinary differential forms, and the exterior product $\wedge$ between such forms is the same as the ordinary exterior product. Moreover, the notation has been chosen so that if α is a double form of type (p, q) and $X_1, \ldots, X_p \in \mathfrak{X}(M)$, then $\alpha(X_1, \ldots, X_p)$ is an ordinary differential form of degree q. Consequently, the definition (4.2) of the exterior product of double forms can be rewritten as

$$(\alpha \wedge \beta)(X_1, \ldots, X_{p+r}) = \sum_{\rho \in \mathrm{Sh}(p,r)} \varepsilon_\rho \, \alpha(X_{\rho_1}, \ldots, X_{\rho_p}) \wedge \beta(X_{\rho_{p+1}}, \ldots, X_{\rho_{p+r}})$$

$$= \frac{1}{p!r!} \sum_{\rho \in \mathfrak{S}_{p+r}} \varepsilon_\rho \, \alpha(X_{\rho_1}, \ldots, X_{\rho_p}) \wedge \beta(X_{\rho_{p+1}}, \ldots, X_{\rho_{p+r}}), \tag{4.3}$$

In computations, frequently (4.3) is easier to work with than (4.2).

In the context of Riemannian manifolds the two most important symmetric double forms are the metric tensor field $g = \langle \ , \ \rangle$ and the curvature tensor R. Clearly, g has type $(1, 1)$ and R has type $(2, 2)$. Then the wedge product of g with itself c times, which we denote by g^c, is a double form of type (c, c); similarly, R^c is a double form of type $(2c, 2c)$. Although the definition (4.2) is rather complicated in the most general case, it is possible to give inductive formulas for g^c and R^c. These inductive formulas are special cases of the following lemma.

Lemma 4.1. *Let ψ and Ψ be double forms of type $(1, 1)$ and $(2, 2)$, respectively. Then*

$$\psi^c(X_1, \ldots, X_c)(Y_1, \ldots, Y_c)$$

$$= \sum_{ij=1}^{c} (-1)^{i+j} \psi(X_i)(Y_j) \psi^{c-1}(X_1, \ldots, \hat{X}_i, \ldots, X_c)(Y_1, \ldots, \hat{Y}_j, \ldots, Y_c),$$

and

$$\Psi^c(X_1,\dots,X_{2c})(Y_1,\dots,Y_{2c}) = \sum_{\substack{1\le i<j\le 2c\\ 1\le k<l\le 2c}} (-1)^{i+j+k+l}\Psi(X_i,X_j)(Y_k,Y_l)$$
$$\cdot\Psi^{c-1}(X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{2c})(Y_1,\dots,\hat{Y}_k,\dots,\hat{Y}_l,\dots,Y_{2c}).$$

for $X_1,\dots,X_{2c},Y_1,\dots,Y_{2c}\in\mathfrak{X}(M)$.

Proof. These formulas follow by induction from the definition of double form. □

It will also be necessary to consider the **contraction operators** C^c on the space of double forms of type (p,q) of a Riemannian manifold M. These operators are defined inductively by $C^0(\alpha)=\alpha$ and

$$C^c(\alpha)(X_1,\dots,X_{p-c})(Y_1,\dots,Y_{q-c}) =$$
$$\sum_{a=1}^{n} C^{c-1}(\alpha)(X_1,\dots,X_{p-c},E_a)(Y_1,\dots,Y_{q-c},E_a),$$

where $\{E_1,\dots,E_n\}$ is any orthonormal local frame field on M. (It is easy to show that this definition does not depend on the choice of frame field.)

We can now explain the meaning of all the coefficients in Weyl's Tube Formula (1.1).

Definition. *Let R denote the curvature tensor of a Riemannian manifold P of dimension q. We put*

$$k_{2c}(P) = \frac{1}{c!(2c)!}\int_P C^{2c}(R^c)\,dP. \tag{4.4}$$

We call $k_{2c}(P)$ the $(2c)^{\text{th}}$ **integrated mean curvature** *of P.*

In order for this definition to make sense, it is, of course, necessary that the integral on the right-hand side of (4.4) converge. This is the case, for example, if P is compact, or more generally, if P has compact closure. The name "integrated mean curvature" will become clear in Chapter 10, where we also define the odd integrated mean curvatures.

We work out the first three of these coefficients explicitly. In the following lemma, and in fact in the rest of the book, it will be convenient to use the notation R_{abcd} as an abbreviation for $R(E_a,E_b)(E_c,E_d)$.

Lemma 4.2. *We have*

$$k_0(P) = \text{volume}(P), \tag{4.5}$$

$$k_2(P) = \frac{1}{2}\int_P \tau\,dP, \tag{4.6}$$

$$k_4(P) = \frac{1}{8}\int_P \{\tau^2 - 4\|\rho\|^2 + \|R\|^2\}\,dP. \tag{4.7}$$

(The terms on the right-hand side of (4.7) will be discussed in the course of the proof.)

Proof. By convention $C^0(R^0) = 1$, so $k_0(P)$ is given by (4.5). Furthermore, $C^2(R^1)$ is just the scalar curvature:

$$C^2(R^1) = \sum_{ab=1}^{q} R(E_a, E_b)(E_a, E_b) = \sum_{ab=1}^{q} R_{abab} = \tau = \tau(R).$$

Thus (4.6) holds.

In order to discuss $k_4(P)$, we need to introduce the **quadratic curvature invariants**. These are the three possible complete contractions of the tensor product of the curvature tensor with itself (see Lemma 4.4). (The scalar curvature, on the other hand, is the unique **linear curvature invariant.**) Thus the quadratic curvature invariants are the square τ^2 of the scalar curvature, and in addition

$$\|\rho\|^2 = \sum_{ab=1}^{q} \left(\sum_{c=1}^{q} R_{acbc} \right)^2 \qquad \text{and} \qquad \|R\|^2 = \sum_{abcd=1}^{q} R_{abcd}^2.$$

Here $\|\rho\|$ is called the **length of the Ricci curvature** ρ, and $\|R\|$ is called the **length of the curvature**.

For the computation of $C^4(R^2)$ in terms of τ^2, $\|\rho\|^2$ and $\|R\|^2$, we first observe that

$$R^2(X_1, X_2, X_3, X_4) = 2 \underset{234}{\mathfrak{S}} R(X_1, X_2) \wedge R(X_3, X_4).$$

Hence

$$\begin{aligned} C^4(R^2) &= \sum_{abcd=1}^{q} R^2(E_a, E_b, E_c, E_d)(E_a, E_b, E_c, E_d) \\ &= 2 \sum_{abcd=1}^{q} \underset{bcd}{\mathfrak{S}} \big\{ R(E_a, E_b) \wedge R(E_c, E_d) \big\} (E_a, E_b, E_c, E_d) \\ &= 6 \sum_{abcd=1}^{q} \big\{ R_{abab} R_{cdcd} - 4 R_{abac} R_{dbdc} + R_{abcd} R_{abcd} \big\} \\ &= 6 \{ \tau^2 - 4\|\rho\|^2 + \|R\|^2 \}, \end{aligned}$$

and so from (4.4) we get (4.7). □

4.2 Invariants

In the last section of [Weyl1], the theory of invariants is used to complete the derivation of his tube formula. Instead of getting involved in the intricacies of this complicated theory, we shall use only those parts of the theory that we need. The

material in the rest of this section is adapted from [BGM, pages 75–76]. Other relevant sources are [ABP], [Ep], [Ku], [Spivak, volume 5, pages 466–486] and [Weyl2].

Let V be an n-dimensional vector space over $\mathbb{R}$ with inner product $\langle\ ,\ \rangle$. Then the orthogonal group $O(n)$ has a natural representation on V, which we write as $v \longmapsto gv$ for $v \in V$ and $g \in O(n)$. Next, let W_k denote the dual space of the k-fold tensor product $V \otimes \cdots \otimes V$; equivalently $W_k = V^* \otimes \cdots \otimes V^*$, where V^* is the dual space of V. Then the natural representation of $O(n)$ on V induces a representation on W_k, which is given by

$$g(\phi)(v_1 \otimes \cdots \otimes v_k) = \phi(g^{-1}v_1 \otimes \cdots \otimes g^{-1}v_k)$$

for $\phi \in W_k$ and $v_1, \ldots, v_k \in V$. Also, the symmetric group $\mathfrak{S}_k$ acts on W_k via the formula

$$\sigma(\phi)(v_1 \otimes \cdots \otimes v_k) = \phi(v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_k}),$$

for $\sigma \in \mathfrak{S}_k$ and $\phi \in W_k$.

By definition a polynomial of degree h on W_k is a mapping $P\colon W_k \longrightarrow \mathbb{R}$ which has a symmetric multilinear extension to the h-fold tensor product $W_k \otimes \overset{h}{\cdots} \otimes W_k$. Let $P_h(W_k)$ denote the space of all such polynomials. Then $O(n)$ also acts on $P_h(W_k)$ via

$$g(P)(\phi) = P(g^{-1}\phi).$$

We are interested in **invariants** of $O(n)$, that is, those elements P of $P_h(W_k)$ such that

$$g(P) = P \quad \text{for} \quad g \in O(n).$$

We now write down examples of nonzero invariants in the case that k is even, say $k = 2m$. For $\phi \in W_{2m}$ and $\sigma \in \mathfrak{S}_{2m}$ we put

$$P_\sigma(\phi) = \sum_{a_1 \ldots a_m = 1}^{n} \sigma(\phi)(e_{a_1}, e_{a_1}, \ldots, e_{a_m}, e_{a_m}), \tag{4.8}$$

where $\{e_1, \ldots, e_n\}$ is any orthonormal basis of V. It is easy to see that this definition does not depend on the choice of orthonormal basis. But we need a generalization of this construction. For this we introduce the following notation: let $\otimes^h \phi \in W_{hk}$ be defined by

$$(\otimes^h \phi)(v_1 \otimes \cdots \otimes v_{hk}) = \phi(v_1 \otimes \cdots \otimes v_k) \cdots \phi(v_{k(h-1)+1} \otimes \cdots \otimes v_{hk})$$

for $\phi \in W_k$ and $v_1, \ldots, v_{hk} \in V$.

Definition. *Suppose that hk is even, say $hk = 2m$, and let $\sigma \in \mathfrak{S}_{2m}$. Then the* **elementary invariant** *corresponding to σ is the polynomial $P_\sigma \in P_h(W_k)$ defined by*

$$P_\sigma(\phi) = \sum_{a_1 \ldots a_m = 1}^{n} \sigma(\otimes^h \phi)(e_{a_1}, e_{a_1}, \ldots, e_{a_m}, e_{a_m}), \tag{4.9}$$

for $\phi \in W_k$.

Note that the P_σ in (4.8) is linear in ϕ, whereas the P_σ in (4.9)) is a polynomial of degree h in ϕ. It is easy to see that the right-hand side of (4.9) is independent of the choice of orthonormal basis. It follows that every elementary invariant P_σ satisfies $g(P_\sigma) = P_\sigma$ for all $g \in O(n)$.

The most important fact from invariant theory that we shall need is the next theorem.

Theorem 4.3. *Every invariant polynomial is a sum of products of the elementary invariants.*

This key theorem from the theory of invariants is beyond the scope of this book. For an indication of the proof see [BGM, page 76] and [Spivak, volume 5, page 481]. It is a theorem about $O(n)$. For $SO(n)$ the space of invariant polynomials is generated by the elementary invariants together with the determinant.

Let U be a subspace of W_k. Instead of studying the invariant polynomials on W_k we can study the invariant polynomials on U. The space of such polynomials is generated by restrictions of elementary invariants to U. However, the restrictions of linearly independent polynomials may be linearly dependent.

An important application of Theorem 4.3 is the classification of quadratic invariants of curvature tensors. We take V to be the tangent space M_m to a Riemannian manifold M and $k = 4$. However, instead of using all of the dual space of $V \otimes V \otimes V \otimes V$, we take the subspace $\mathfrak{R}$ of curvature tensors, that is, all multilinear mappings $R\colon V \otimes V \otimes V \otimes V \longrightarrow \mathbb{R}$ that satisfy the identities (2.16)–(2.19). Obviously, every polynomial on the dual space of $V \otimes V \otimes V \otimes V$ restricts to a polynomial on $\mathfrak{R}$, although some may be zero there.

More generally, we can consider contractions on $\otimes^k\mathfrak{R}$, which is a subspace of the dual space of $\otimes^{4k}V$. Thus if $R_1, \ldots, R_k \in \mathfrak{R}$, then

$$R_1 \otimes \cdots \otimes R_k\colon \otimes^{4k}V \longrightarrow \mathbb{R}$$

is an element of $\otimes^k\mathfrak{R}$. It can be contracted in many different ways. A **complete contraction** on $\otimes^k\mathfrak{R}$ is a linear operator that assigns to each element $A \in \otimes^k\mathfrak{R}$ the real number that is the contraction over all the arguments of A taken in some order.

This leads us to make the following definition:

Definition. *A* **curvature invariant of order** k *is a complete contraction of an element of the form* $R \otimes \cdots \otimes R$ *in the space* $\otimes^k\mathfrak{R}$.

Thus, for example, a quadratic invariant of a curvature tensor R is a complete contraction of $R \otimes R \in \mathfrak{R}^2$. We now describe some of the low order curvature invariants.

Lemma 4.4. *The curvature invariants of orders* 1 *and* 2 *are as follows.*

(i) *The scalar curvature τ is the only linear curvature invariant.*

(ii) *The quadratic curvature invariants*[2] *are precisely*

$$\tau^2, \quad \|\rho\|^2 \text{ and } \|R\|^2.$$

Proof. Part (i) is easy, so we prove part (ii). The mapping $R \otimes R$ is a function of eight variables. Let us count the different ways to contract $R \otimes R$ four times. Each contraction involves two indices. Three possibilities for contractions are exemplified by τ^2, $\|\rho\|^2$ and $\|R\|^2$. To get τ^2, each pair of contracting indices must occur entirely in the first factor or entirely in second factor of $R \otimes R$, whereas for $\|R\|^2$ each pair must be split over the two factors. On the other hand, of the four pairs in the definition of $\|\rho\|^2$, two are split, one pair is in the first factor, and one pair is in the second factor.

We can use an element $\sigma \in \mathfrak{S}_8$ to permute the various indices of $R \otimes R$ before carrying out the contractions. So by definition τ^2, $\|\rho\|^2$ and $\|R\|^2$ are curvature invariants. Since $h = 2$ in the definition of elementary invariant, they are quadratic invariants. In fact, using the first Bianchi identity (2.19) one sees that on the space $\mathfrak{R}$ any other quadratic invariant must be a linear combination of τ^2, $\|\rho\|^2$ and $\|R\|^2$. The most complicated case involves the verification that

$$\sum_{abcd=1}^{n} R_{abcd} R_{acbd} = \frac{1}{2}\|R\|^2. \qquad \square$$

4.3 Moments and Invariants

In this section we shall use the theory of invariants (more precisely, we shall use Theorem 4.3) to express moments as finite sums. (See Section A.A.2 of the Appendix for some of the elementary properties of moments and [ST] for a historical discussion of moments.)

Let V be an n-dimensional vector space with positive definite inner product $\langle\ ,\ \rangle$. Define $I_s \colon \otimes^s V^* \longrightarrow \mathbb{R}$ by

$$I_s(\phi) = \int_{S^{n-1}(1)} \phi(v, \ldots, v)\, dv,$$

where $S^{n-1}(1)$ denotes the unit sphere in V defined by the inner product. Note that $I_s \in P_1(W_s)$.

[2]There is a more general definition of curvature invariant that takes into account not only the contractions of $\mathfrak{R}$, but also the covariant derivatives of the curvature tensor. Such generalized curvature invariants are not needed for the proof of Weyl's Tube Formula, but they are important for the study of the heat equation, and also for the study of power series expansion for the volume of a geodesic ball that we shall give in Chapter 9. There is one additional quadratic curvature invariant in this generalized sense, namely $\Delta\tau$. See problems 4.7 and 4.8.

Lemma 4.5. *$I_s = 0$ for s odd, and for $\phi \in \otimes^{2s}V^* = W_{2s}$ we have*

$$I_{2s}(\phi) = \frac{2\pi^{n/2}}{n(n+2)\cdots(n+2s-2)\Gamma(\frac{n}{2})} \sum_{\sigma\in\mathfrak{Q}_s} \sum_{i_1\ldots i_s=1}^{n} \sigma(\phi)(e_{i_1}, e_{i_1}, \ldots, e_{i_s}, e_{i_s}), \tag{4.10}$$

where

$$\mathfrak{Q}_s = \left\{ \sigma \in \mathfrak{S}_{2s} \mid \sigma_1 < \sigma_3 < \cdots < \sigma_{2s-1} \text{ and } \sigma_{2t-1} < \sigma_{2t} \text{ for } t = 1, \ldots, s \right\},$$

and $\{e_1, \ldots, e_n\}$ is any orthonormal basis of V.

Proof. Theorem 4.3 implies that $I_s = 0$ for s odd, and that I_{2s} is a linear combination of elementary invariants. So after changing notation, we can write

$$I_{2s}(\phi) = \sum_{\sigma\in\mathfrak{S}_s} b_\sigma\left(\sum_{i_1\ldots i_s=1}^{n} \sigma(\phi)(e_{i_1}, e_{i_1}, \ldots, e_{i_s}, e_{i_s}) \right). \tag{4.11}$$

To find the b_σ's, it suffices to evaluate both sides of (4.11) on some well chosen ϕ; for example, we take $\phi = u_{i_1}^2 \cdots u_{i_s}^2$. It follows from Theorem 1.5 of the Appendix (page 251) that for any $\rho \in \mathfrak{S}_{2s}$ we have

$$\begin{aligned}
\frac{2\pi^{n/2}}{n(n+2)\cdots(n+2s-2)\Gamma(\frac{n}{2})} &= I_{2s}(u_{i_1}^2 \cdots u_{i_s}^2) = I_{2s}(\rho^{-1}(u_{i_1}^2 \cdots u_{i_s}^2)) \\
&= \sum_{\sigma\in\mathfrak{S}_s} b_\sigma\left(\sum_{i_1\ldots i_s=1}^{n} \sigma(\rho^{-1}(u_{i_1}^2 \cdots u_{i_s}^2))(e_{i_1}, e_{i_1}, \ldots, e_{i_s}, e_{i_s}) \right) \\
&= \sum_{\sigma\in\mathfrak{S}_s} b_{\sigma\rho}\left(\sum_{i_1\ldots i_s=1}^{n} \sigma(u_{i_1}^2 \cdots u_{i_s}^2)(e_{i_1}, e_{i_1}, \ldots, e_{i_s}, e_{i_s}) \right) \\
&= 2^s s!\, b_\rho.
\end{aligned}$$

Thus all the b_σ's are equal; in fact,

$$b_\sigma = \frac{2\pi^{n/2}}{n(n+2)\cdots(n+2s-2)\Gamma(\frac{n}{2})2^s s!} \tag{4.12}$$

for all $\sigma \in \mathfrak{S}_{2s}$. Hence we obtain (4.10) from (4.11) and (4.12). ∎

An element $\phi \in \otimes^{2s}V^*$ is called **symmetric** provided $\sigma(\phi) = \phi$ for all $\sigma \in \mathfrak{S}_{2s}$. There is an important simplification of formula (4.10) for a symmetric ϕ.

Corollary 4.6. *If $\phi \in \otimes^{2s}V^*$ is symmetric, then*

$$I_{2s}(\phi) = \frac{1 \cdot 3 \cdots (2s-1) \cdot 2\pi^{n/2}}{n(n+2)\cdots(n+2s-2)\Gamma(\frac{n}{2})} \sum_{i_1 \ldots i_s = 1}^{n} \phi(e_{i_1}, e_{i_1}, \ldots, e_{i_s}, e_{i_s}). \tag{4.13}$$

Proof (Making use of Weyl's theory of invariants). If ϕ is symmetric, then all the terms on the right-hand side of (4.10) are all the same. The cardinality of $\mathfrak{Q}_{2s}$ is

$$\frac{(2s)!}{2^s s!} = 1 \cdot 3 \cdots (2s-1),$$

and so (4.13) follows from (4.10). □

Second Proof (Independent of Weyl's theory of invariants). Both the left and right-hand sides of (4.13) are linear in ϕ. By Theorem 1.5 of the Appendix (page 251), equation (4.13) holds for any ϕ of the form

$$\phi = u_1^{i_1} \cdots u_s^{i_s}, \tag{4.14}$$

where $i_1, \ldots, i_s$ are any even integers. Since any *symmetric* $\phi \in \otimes^{2s}V^*$ is the sum of polynomials of the form of the right-hand side of (4.14), it follows that (4.13) holds for any symmetric $\phi \in \otimes^{2s}V^*$. □

4.4 Averaging the Tube Integrand

In this section we show how to express the tube integrand entirely in terms of the curvature tensor. Then we complete the proof of Weyl's Tube Formula.

The key observation of Weyl is that although $t \longmapsto \det(\delta_{ab} - tT_{abu})$ depends on the second fundamental form T (and hence the embedding of P in M), integration over all the unit vectors u gets rid of this dependence. In other words,

$$\int_{S^{n-q-1}(1)} \det(\delta_{ab} - tT_{abu})\, du$$

is expressible in terms of the curvature of P alone. The proof of this fact is unfortunately rather complicated. We tackle it in the next theorem.

Theorem 4.7. *Suppose that $P \subset \mathbb{R}^n$ is a q-dimensional submanifold with second fundamental form T and curvature tensor R^P. Let $p \in P$ and let $S^{n-q-1}(1)$ denote*

the unit sphere in $P_p^\perp$. *Then*[3]

$$\int_{S^{n-q-1}(1)} \det(\delta_{ab} - t\,T_{abu})\,du \tag{4.15}$$

$$= \frac{2\pi^{(n-q)/2}}{\Gamma(\frac{1}{2}(n-q))} \sum_{c=0}^{[q/2]} \frac{C^{2c}\big((R^P)^c\big)_p t^{2c}}{c!(2c)!(n-q)(n-q+2)\cdots(n-q+2c-2)}$$

$$= 2\pi^{(n-q)/2} \sum_{c=0}^{[q/2]} \frac{C^{2c}\big((R^P)^c\big)_p t^{2c}}{c!(2c)!\Gamma(\frac{1}{2}(n-q)+c)2^c}.$$

Proof. First, let ψ_c be defined by

$$\psi_c(u_1,\ldots,u_c) = \frac{1}{(c!)^2} \sum_{a_1\ldots a_c=1}^{q} \sum_{\sigma,\pi\in\mathfrak{S}_c} \varepsilon_\sigma\,\varepsilon_\pi\,T_{a_{\sigma(1)}a_{\pi(1)}u_1}\cdots T_{a_{\sigma(c)}a_{\pi(c)}u_c} \tag{4.16}$$

for $u_1,\ldots,u_c \in P_p^\perp$. Notice that ψ_c is symmetric. The rule for expanding the determinant of the matrix $\det(\delta_{ab} - tT_{abu})$ by minors can be written as

$$\det(\delta_{ab} - t\,T_{abu}) = \sum_{c=0}^{q} \psi_c(u,\ldots,u)t^c \tag{4.17}$$

for $u \in P_p^\perp$. We must integrate each ψ_c over $S^{n-q-1}(1)$. The ψ_{2c+1}'s integrate to zero because integration over one hemisphere is cancelled by the integration over the other. For the ψ_{2c}'s we have by Corollary 4.6 that

$$\begin{aligned} I_{2c}(\psi_{2c}) &= \frac{1\cdot 3\cdots(2c-1)\cdot 2\pi^{(n-q)/2}\sum_{i_1\ldots i_c=1}^{n}\psi_{2c}(e_{i_1},e_{i_1},\ldots,e_{i_c},e_{i_c})}{(n-q)(n-q+2)\cdots(n-q+2c-2)\Gamma(\frac{1}{2}(n-q))} \\ &= \frac{(2c)!\,2\pi^{(n-q)/2}}{c!4^c\Gamma(\frac{1}{2}(n-q)+c)} \sum_{i_1\ldots i_c=1}^{n} \psi_{2c}(e_{i_1},e_{i_1},\ldots,e_{i_c},e_{i_c}). \end{aligned} \tag{4.18}$$

The **Gauss equation** (see, for example, [ON4, page 100]) relates the second fundamental form of a submanifold P with the curvature tensors of P and the manifold into which it is embedded. In the case that $P \subset \mathbb{R}^n$ the Gauss equation is

$$\begin{aligned} R^P_{abcd} &= \sum_{i=q+1}^{n} (T_{aci}T_{bdi} - T_{adi}T_{bci}) \\ &= \langle T_{ac}, T_{bd}\rangle - \langle T_{ad}, T_{bc}\rangle. \end{aligned} \tag{4.19}$$

[3]Interpret $(n-q)(n-q+2)\cdots(n-q+2c-2)$ as 1 when $c=0$.

We calculate the right-hand side of (4.18) making use of (4.19); from (4.16) we find that

$$\sum_{i_1\dots i_c=q+1}^{n} \psi_{2c}(e_{i_1}, e_{i_1}, \dots, e_{i_c}, e_{i_c}) \tag{4.20}$$

$$= \frac{1}{((2c)!)^2} \sum_{i_1\dots i_c=q+1}^{n} \sum_{\sigma,\pi\in\mathfrak{S}_{2c}} \sum_{a_1\dots a_{2c}=1}^{q} \varepsilon_\sigma\varepsilon_\pi \Big\{ T_{a_{\sigma(1)}a_{\pi(1)}i_1} T_{a_{\sigma(2)}a_{\pi(2)}i_1} \cdots T_{a_{\sigma(2c-1)}a_{\pi(2c-1)}i_c} T_{a_{\sigma(2c)}a_{\pi(2c)}i_c} \Big\}.$$

Since $\{e_{q+1}, \dots, e_n\}$ is an orthonormal basis for $P_p^\perp$, we have

$$\sum_{i=q+1}^{n} T_{a_{\sigma(s)}a_{\pi(t)}i} T_{a_{\sigma(u)}a_{\pi(v)}i} = \langle T_{a_{\sigma(s)}a_{\pi(t)}}, T_{a_{\sigma(u)}a_{\pi(v)}} \rangle.$$

Thus the right-hand side of (4.20) can be written as

$$\frac{1}{((2c)!)^2} \sum_{a_1\dots a_{2c}=1}^{q} \sum_{\sigma,\pi\in\mathfrak{S}_{2c}} \varepsilon_\sigma\, \varepsilon_\pi \Big\{ \langle T_{a_{\sigma(1)}a_{\pi(1)}}, T_{a_{\sigma(2)}a_{\pi(2)}} \rangle \cdots \langle T_{a_{\sigma(2c-1)}a_{\pi(2c-1)}}, T_{a_{\sigma(2c)}a_{\pi(2c)}} \rangle \Big\}. \tag{4.21}$$

Next, we write (4.21) in a way that at first sight seems more complicated:

$$\frac{1}{2^c((2c)!)^2} \sum_{a_1\dots a_{2c}=1}^{q} \sum_{\sigma,\pi\in\mathfrak{S}_{2c}} \varepsilon_\sigma\, \varepsilon_\pi \Big\{ \Big(\langle T_{a_{\sigma(1)}a_{\pi(1)}}, T_{a_{\sigma(2)}a_{\pi(2)}} \rangle - \langle T_{a_{\sigma(1)}a_{\pi(2)}}, T_{a_{\sigma(2)}a_{\pi(1)}} \rangle \Big) \cdots \Big(\langle T_{a_{\sigma(2c-1)}a_{\pi(2c-1)}}, T_{a_{\sigma(2c)}a_{\pi(2c)}} \rangle - \langle T_{a_{\sigma(2c-1)}a_{\pi(2c)}}, T_{a_{\sigma(2c)}a_{\pi(2c-1)}} \rangle \Big) \Big\}. \tag{4.22}$$

But each of the factors in (4.22) is expressible in terms of the curvature tensor R^P making use of the Gauss equation (4.19). Thus from (4.19)–(4.22) we obtain

$$\sum_{i_1\dots i_c=q+1}^{n} \psi_{2c}(e_{i_1}, e_{i_1}, \dots, e_{i_c}, e_{i_c}) \tag{4.23}$$

$$= \frac{1}{2^c((2c)!)^2} \sum_{a_1\dots a_{2c}=1}^{q} \sum_{\sigma,\pi\in\mathfrak{S}_{2c}} \varepsilon_\sigma\varepsilon_\pi \Big\{ R^P_{a_{\sigma(1)}a_{\sigma(2)}a_{\pi(1)}a_{\pi(2)}} \cdots R^P_{a_{\sigma(2c-1)}a_{\sigma(2c)}a_{\pi(2c-1)}a_{\pi(2c)}} \Big\}$$

$$= \frac{2^c}{((2c)!)^2} \sum_{a_1 \dots a_{2c}=1}^{q} (R^P)^c(a_1, \dots, a_{2c})(a_1, \dots, a_{2c})$$

$$= \frac{2^c}{((2c)!)^2} C^{2c}\big((R^P)^c\big).$$

Now we get a nice formula for the integral of ψ_{2c} over the unit sphere in $P_p^{\perp}$. From (4.18) and (4.23) it follows that

$$I_{2c}(\psi_{2c}) = \frac{2\pi^{(n-q)/2} C^{2c}\big((R^P)^c\big)}{c!(2c)!2^c \Gamma\big(\frac{1}{2}(n-q)+c\big)}. \tag{4.24}$$

When we combine (4.17) and (4.24), we obtain

$$\int_{S^{n-q-1}(1)} \det(\delta_{ab} - tT_{abu})\, du = \sum_{c=0}^{q} (-t)^c \int_{S^{n-q-1}(1)} \psi_c(u, \dots, u)\, du$$

$$= \sum_{c=0}^{[q/2]} I_{2c}(\psi_{2c}) = 2\pi^{(n-q)/2} \sum_{c=0}^{[q/2]} \frac{C^{2c}\big((R^P)^c\big)_p t^{2c}}{c!(2c)!\Gamma\big(\frac{1}{2}(n-q)+c\big)2^c}.$$

Thus we get (4.15). □

At last we are able to complete the proof of Weyl's Tube Formula (1.1).

Theorem 4.8. (Weyl's Tube Formula.) *Let P be a q-dimensional topologically embedded submanifold in Euclidean space $\mathbb{R}^n$. Assume that P has compact closure, and that every point in the tube $T(P, r)$ has a unique shortest geodesic connecting it with P. Then the volume $V_P^{\mathbb{R}^n}(r)$ of $T(P, r)$ is given by*

$$V_P^{\mathbb{R}^n}(r) = \frac{(\pi r^2)^{(n-q)/2}}{\big(\frac{1}{2}(n-q)\big)!} \sum_{c=0}^{[q/2]} \frac{k_{2c}(P) r^{2c}}{(n-q+2)(n-q+4)\cdots(n-q+2c)}. \tag{4.25}$$

Proof. We multiply (4.15) by t^{n-q-1} and integrate over P. By Lemma 3.13 and Theorem 3.15 the left-hand side of the resulting equation is $A_P^{\mathbb{R}^n}(t)$. For the right-hand side we make use of the definition (4.4) of the $k_{2c}(P)$'s in terms of the $C^{2c}(R^P)$'s. The result is

$$A_P^{\mathbb{R}^n}(t) = \frac{2\pi^{(n-q)/2}}{\Gamma\big(\frac{1}{2}(n-q)\big)} \sum_{c=0}^{[q/2]} \frac{k_{2c}(P) t^{n-q-1+2c}}{(n-q)(n-q+2)\cdots(n-q+2c-2)} \tag{4.26}$$

$$= \frac{\pi^{(n-q)/2}}{\big(\frac{1}{2}(n-q)\big)!} \sum_{c=0}^{[q/2]} \frac{k_{2c}(P) t^{n-q-1+2c}}{(n-q+2)\cdots(n-q+2c-2)}.$$

When (4.26) is integrated from 0 to r, we get (4.25). □

Finally,

Corollary 4.9. *The tube volume* $V_P^{\mathbb{R}^n}(r)$ *depends only on* P *and* r *and not on the particular way that* P *is embedded in* $\mathbb{R}^n$.

Proof. Since the coefficients in (4.25) depend only on the curvature of P, they, and also the tube volume, are intrinsic to P. □

Historical Remarks. A key result needed to establish Weyl's Tube Formula is Theorem 4.7. A slight generalization of this theorem is given in [AW]. Then in a footnote Allendoerfer and Weil remark

> Similar calculations may also be found in W. Killing, **Die nicht-euklidischen Raumformen in analytischer Behandlungen,** Teubner, Leipzig, 1885, p. 255.

In a set of lecture notes [Al2, page 15] Allendoerfer is more explicit. He first writes down the tube formula and then says

> The integration over S^{n-1} can be carried through explicitly (see H. Weyl – Am. Journ. of Math. 1939, although the results are actually due to Killing).

All of this may be true, but a great deal of work would be needed to obtain a tube formula from Killing's[4] calculations.

It should also be mentioned that Weyl's derivation of his formula (1.1) is somewhat different from the one presented here. He invokes the theory of invariants at the crucial point, but before that he relies heavily on explicit formulas using the calculus of Euclidean space. See [BeGo, pages 207–243] for more details on this approach. At the same time Weyl derives a formula for tubes in a sphere S^n by observing that S^n is a hypersurface in $\mathbb{R}^{n+1}$ and then translating the formulas for S^n to $\mathbb{R}^{n+1}$ and back again. Besides being a little simpler, the approach we have taken (using the Riccati differential equation for the second fundamental forms) has the advantage that it strongly suggests how to get tube formulas in more general spaces, such as spaces of constant sectional curvature or constant holomorphic sectional curvature. We shall pursue these ideas in the next section, and also later in Chapters 7 and 8.

[4] Wilhelm Karl Joseph Killing (1847–1923). German mathematician. Killing introduced Lie algebras independently of Lie in his study of non-euclidean geometry. The main tools in the classification of the semisimple Lie algebras are Cartan subalgebras and the Cartan matrix, both first introduced by Killing.(But Cartan found some errors in his papers). The notions of root system of a semisimple Lie algebra and characteristic equation of a matrix are due to Killing.

4.5 Generalizations

Corollary 4.9, and hence Weyl's Tube Formula, is not true for a submanifold P of a general Riemannian manifold M. This is easy to prove, for example, when P is a point $m \in M$. We shall see in Chapter 9 that the volume of a geodesic ball in M centered at m has a power series expansion that starts off as

$$V_m^M(r) = \frac{(\pi r^2)^{\frac{n}{2}}}{\left(\frac{n}{2}\right)!}\left\{1 - \frac{\tau^M r^2}{6(n+2)} + O(r^4)\right\}_m,$$

where τ^M denotes the scalar curvature of M. Thus if M has nonconstant scalar curvature, the function $m \longmapsto V_m^M(r)$ will not be constant. In other words, $V_m^M(r)$ depends on where m is embedded in M. However, it is possible to generalize part of the proof of Weyl's Tube Formula, and we shall need this generalization in later chapters.

Let P be a submanifold of a Riemannian manifold M. We use the notation $R^P - R^M$ to denote the restriction to P of the difference of the curvature operators of P and M. The operator $R^P - R^M$ will appear often in the rest of the book because of the Gauss equation (4.28); it is a rough measure of the failure of P to be totally geodesic in M. Although it seems that this operator has something to do with the curvature of the normal bundle of P in M, this is not the case. For example, when M is Euclidean space $R^P - R^M$ and R^P coincide.

The explicit integration of $\vartheta_u(t)$ over unit vectors u is impractical for a general Riemannian manifold M. (It is very complicated, for example, even for a point in the product of two spaces of constant curvature.) However, there is nothing to prevent the integration of $\det(\delta_{ab} - tT_{abu})$ over the unit sphere in $P_p^\perp$.

Theorem 4.10. *Suppose that $P \subset M$ is a q-dimensional submanifold with second fundamental form T on a Riemannian manifold M. For each point $p \in P$ let $S^{n-q-1}(1)$ denote the unit sphere in $P_p^\perp$. Then we have*

$$\int_{S^{n-q-1}(1)} \det(\delta_{ab} - t\,T_{abu})\,du \tag{4.27}$$

$$= \frac{2\pi^{(n-q)/2}}{\Gamma(\frac{1}{2}(n-q))}\sum_{c=0}^{[q/2]} \frac{C^{2c}((R^P - R^M)^c)t^{2c}}{c!(2c)!(n-q)(n-q+2)\cdots(n-q+2c-2)}$$

$$= 2\pi^{(n-q)/2}\sum_{c=0}^{[\frac{q}{2}]} \frac{C^{2c}((R^P - R^M)^c)t^{2c}}{c!(2c)!\Gamma(\frac{1}{2}(n-q)+c)2^c}.$$

Proof. The proof of Theorem 4.10 is exactly the same as that of Theorem 4.7 with one exception, the Gauss equation. Instead of (4.19) we have the more general

formula

$$R^P_{abcd} - R^M_{abcd} = \sum_{i=q+1}^{n} (T_{aci}T_{bdi} - T_{adi}T_{bci}) = \langle T_{ac}, T_{bd}\rangle - \langle T_{ad}, T_{bc}\rangle. \quad (4.28)$$

This is the **Gauss equation** for a submanifold P of a Riemannian manifold M (see [ON4, page 100]). So the proof of Theorem 4.10 is word for word that of Theorem 4.7, except that (4.28) is used instead of (4.19). □

The reason that it is not possible to prove Weyl's Tube Formula for general Riemannian manifolds is that Lemma 3.14 holds only for submanifolds of $\mathbb{R}^n$; everything else works fine. In particular, integration of (4.27) over P yields:

Corollary 4.11. *Let P be a q-dimensional submanifold with compact closure of a Riemannian manifold M. Then*

$$\int_0^r \int_P \int_{S^{n-q-1}(1)} t^{n-q-1} \det(\delta_{ab} - t\,T_{abu})\,du\,dP\,dt \quad (4.29)$$

$$= \frac{(\pi r^2)^{(n-q)/2}}{(\frac{1}{2}(n-q))!} \sum_{c=0}^{[q/2]} \frac{k_{2c}(R^P - R^M) r^{2c}}{(n-q+2)(n-q+4)\cdots(n-q+2c)}.$$

It is an interesting fact that the right-hand side of (4.29) does not depend on the second fundamental form of P in M; still we cannot interpret it directly as a tube volume (but see Section 8.2 of Chapter 8). In later chapters the right-hand side of (4.29) will arise frequently. For example, we shall show in Chapter 8 that $K^M \geq 0$ implies that

$$V_P^M(r) \leq \frac{(\pi r^2)^{(n-q)/2}}{(\frac{1}{2}(n-q))!} \sum_{c=0}^{[q/2]} \frac{k_{2c}(R^P - R^M) r^{2c}}{(n-q+2)(n-q+4)\cdots(n-q+2c)}.$$

4.6 Problems

4.1 Using problem 3.5, show that if P is any submanifold in a space $\mathbb{K}^n(\lambda)$ of constant curvature λ, then the volume of a tube of small radius r about P is given by

$$\frac{d}{dr} V_P^{\mathbb{K}^n(\lambda)}(r) = A_P^{\mathbb{K}^n(\lambda)}(r) = \frac{2\pi^{(n-q)/2}}{\Gamma(\frac{1}{2}(n-q))} \left(\frac{\tan(r\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-q-1}$$

$$\cdot \left(\cos(r\sqrt{\lambda})\right)^{n-1} \sum_{c=0}^{[\frac{q}{2}]} \frac{k_{2c}(R^P - \lambda I)\left(\dfrac{\tan(r\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{2c}}{(n-q)(n-q+2)\cdots(n-q+2c-2)}.$$

4.2 Let ψ be a double form of type $(1,1)$. Show that

$$\psi^p(X_1,\ldots,X_p)(Y_1,\ldots,Y_p) = p!\det\big(\psi(X_i)(Y_j)\big)$$

for $X_1,\ldots,X_p,Y_1,\ldots,Y_p \in \mathfrak{X}(M)$, and that

$$\psi^p(X_1,\ldots,X_p) = p!\,\psi(X_1)\wedge\ldots\wedge\psi(X_p).$$

4.3 Let Ψ be a double form of type $(2,2)$. Show that

$$\Psi^p(X_1,\ldots,X_{2p}) = \frac{1}{2^p}\sum_{\rho\in\mathfrak{S}_{2p}} \varepsilon_\rho\,\Psi(X_{\rho_1},X_{\rho_2})\wedge\ldots\wedge\Psi(X_{\rho_{2p-1}},X_{\rho_{2p}})$$

for $X_1,\ldots,X_{2p} \in \mathfrak{X}(M)$.

4.4 Show for a space $\mathbb{K}^n(\lambda)$ of constant curvature λ that the lengths of the Ricci curvature and the curvature tensor are given by

$$\|\rho\|^2 = n(n-1)^2\lambda^2 \quad\text{and}\quad \|R\|^2 = 2n(n-1)\lambda^2.$$

4.5 Show that if R^c is considered to be a linear transformation

$$R^c\colon \Lambda^{2c}(P_p) \longrightarrow \Lambda^{2c}(P_p)$$

then $C^{2c}(R^c) = (2c)!\,\mathrm{tr}(R^c)$.

4.6 Let $J_n(z)$ denote the Bessel function of order n. Using problem 4.5 and the power series expansion

$$\left(\frac{2}{\sqrt{-1}\,z}\right)^{(n-q)/2} J_{(n-q)/2}(\sqrt{-1}\,z) = \sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}z\right)^{2k}}{k!\,\left(\frac{1}{2}(n-q)+k\right)!},$$

show that Weyl's tube formula (1.1) can be rewritten as

$$V_P^{\mathbb{R}^n}(r) = \int_P \mathrm{tr}\left(\left(\frac{\pi}{\sqrt{-1}\,R}\right)^{(n-q)/2} J_{(n-q)/2}(\sqrt{-1}r^2R)\right)dP.$$

(See [Gr15].)

4.7 One can also consider the invariants of the space $\nabla^p\mathfrak{R}$ of covariant derivatives of order p of curvature tensors. Show that $\nabla^2\mathfrak{R}$ has the invariant

$$\Delta\tau = \sum_{acd=1}^{n}\nabla^2_{aa}R_{cdcd}.$$

Usually the $\Delta\tau$ is also considered a quadratic curvature invariant. (For the definition of the covariant derivative of the curvature tensor see equation (6.22), page 94.)

4.8 Show that for a Riemannian manifold of dimension at least 6 the space of cubic curvature invariants has dimension 17. By definition a **cubic invariant** is a complete contraction of one of the spaces

$$\mathfrak{R}^3, \quad (\nabla\mathfrak{R})^2, \quad \nabla^2\mathfrak{R}\otimes\mathfrak{R}, \quad \nabla^4\mathfrak{R}.$$

The following is a specific basis:

$$\tau^3,\ \ \tau\|\rho\|^2,\ \ \tau\|R\|^2,\ \ \tau\Delta\tau,\ \ \Delta^2\tau,\ \ \langle\Delta R, R\rangle = \sum R_{ijkl}\nabla^2_{pp}R_{ijkl},$$

$$\check{\rho} = \sum \rho_{ij}\rho_{jk}\rho_{ki},\ \ \check{R} = \sum R_{ijkl}R_{klpq}R_{pqij},\ \ \check{\check{R}} = \sum R_{ikjl}R_{kplq}R_{piqj},$$

$$\|\nabla\tau\|^2 = \sum(\nabla_i\tau)^2,\ \ \|\nabla\rho\|^2 = \sum(\nabla_i\rho_{jk})^2,$$

$$\alpha(\rho) = \sum(\nabla_i\rho_{jk})(\nabla_k\rho_{ij}),\ \ \|\nabla R\|^2 = \sum(\nabla_i R_{jklq})^2,$$

$$\langle\Delta\rho,\rho\rangle = \sum \rho_{ij}\nabla^2_{kk}\rho_{ij},\ \ \left\langle\nabla^2\tau,\rho\right\rangle = \sum(\nabla^2_{ij}\tau)\rho_{ij},$$

$$\left\langle\rho\otimes\rho,\bar{R}\right\rangle = \sum\rho_{ij}\rho_{kl}\bar{R}_{ijkl} \qquad \text{where } \bar{R}_{ijkl} = R_{ikjl},$$

$$\left\langle\rho,\dot{R}\right\rangle = \sum\rho_{ij}\dot{R}_{ij} \qquad \text{where } \dot{R}_{ij} = \textstyle\sum_{pqr} R_{ipqr}R_{jpqr}.$$

Chapter 5

The Generalized Gauss-Bonnet Theorem

In this chapter we shall prove the Generalized Gauss-Bonnet Theorem using tubes. The principal ingredients are: (1) H. Hopf's generalization [Hopf1], [Hopf2] of the Gauss-Bonnet Theorem for hypersurfaces in $\mathbb{R}^n$, (2) the Nash Embedding Theorem [Nash], (3) Weyl's Tube Formula [Weyl1], and (4) some elementary calculations with volumes of tubes and Euler characteristics. This proof using tubes is neither the most direct nor the most elegant; Chern's proof [Chern1] excels in both of these respects (see the end of this chapter). However, the proof via steps (1)–(4) has many interesting features; it is due to Allendoerfer [Al1] and Fenchel [Fl2]. Furthermore, some of the techniques of the present chapter will be useful when we discuss complex versions of the Weyl Tube Formula in Chapters 6 and 7.

From Weyl's Tube Formula (1.1), we see that the volume $V_P^{\mathbb{R}^n}(r)$ of a tube of radius r about a submanifold $P \subset \mathbb{R}^n$ is a polynomial. In the course of the computations certain coefficients, the $k_{2c}(P)$'s, appear. We know by Theorem 4.8 that these coefficients depend only on the Riemannian metric of P, and not on the particular way in which P is embedded in $\mathbb{R}^n$.

Although the $k_{2c}(P)$'s are independent of the embedding, it would be too much to expect them to be independent as well of the particular metric on P. As a matter of fact, $k_0(P) = \text{volume}(P)$ can change when the metric of P changes. Therefore, it is more than a little surprising that when P is compact and even dimensional, not only is the top coefficient in the Weyl Tube Formula a metric invariant, but also a topological invariant. We shall see that this fact is equivalent to the Generalized Gauss-Bonnet Theorem.

In Section 5.1 we exploit the fact that tubular hypersurfaces about tubular hypersurfaces about a submanifold P are themselves tubular hypersurfaces about P. Elementary facts about the Euler characteristic and the Pfaffian are recalled in Sections 5.2 and 5.3. The proof of the Generalized Gauss-Bonnet Theorem for

hypersurfaces in $\mathbb{R}^{2n+1}$ is outlined in Section 5.4. Then in Section 5.5 we give the proof using tubes of the Generalized Gauss-Bonnet Theorem. Some of the main events in the history of the Gauss-Bonnet Theorem are mentioned in Section 5.6.

5.1 Tubes around Tubular Hypersurfaces

Let P be a topologically embedded submanifold with compact closure in $\mathbb{R}^n$. Each tubular hypersurface P_t about P is also a candidate for a submanifold about which we can consider a tube. Instead of just computing the function

$$t \longmapsto \text{volume } (P_t) = k_0(P_t)$$

(which we already know is the derivative of $V_P^{\mathbb{R}^n}(r)$ at $r = t$), why not compute the functions

$$t \longmapsto k_{2c}(P_t)?$$

In fact, this can be done in an elementary way. Let us assume that P has even dimension, say $2p$, and that it is embedded in an odd dimensional Euclidean space $\mathbb{R}^{2n+1}$. (This will be the case needed for the Gauss-Bonnet Theorem.) Each tubular hypersurface P_t then has dimension $2n$.

Theorem 5.1. *Let $P \subset \mathbb{R}^{2n+1}$ have even dimension $2p$. Then for $a = 1, \ldots, n$ the tube coefficient $k_{2a}(P_t)$ of the tubular hypersurface P_t is given by*

$$\frac{k_{2a}(P_t)}{1 \cdot 3 \cdots (2a+1)} \tag{5.1}$$

$$= \frac{\pi^{n-p+\frac{1}{2}}}{(n-p+\frac{1}{2})!} \sum_{b=0}^{n-a} \frac{k_{2(b+a-n+p)}(P)}{(2n-2p+3)\cdots(2a+2b+1)} \binom{2a+2b+1}{2a+1} t^{2b}.$$

Proof. We apply Weyl's Tube Formula (1.1) to the hypersurface P_t. Since P_t has codimension 1, we obtain

$$V_{P_t}^{\mathbb{R}^{2n+1}}(r) = \frac{(\pi r^2)^{1/2}}{(\frac{1}{2})!} \sum_{c=0}^{n} \frac{k_{2c}(P_t) r^{2c}}{1 \cdot 3 \cdots (2c+1)} \tag{5.2}$$

$$= 2 \sum_{c=0}^{n} \frac{k_{2c}(P_t) r^{2c+1}}{1 \cdot 3 \cdots (2c+1)}.$$

On the other hand, we can express the left-hand side of (5.2) as the difference of volumes of tubes about P:

$$V_{P_t}^{\mathbb{R}^{2n+1}}(r) = V_P^{\mathbb{R}^{2n+1}}(t+r) - V_P^{\mathbb{R}^{2n+1}}(t-r). \tag{5.3}$$

We expand the right-hand side of (5.3), again making use of Weyl's Tube Formula (1.1):

$$V_P^{\mathbb{R}^{2n+1}}(t+r) - V_P^{\mathbb{R}^{2n+1}}(t-r)$$

$$= \frac{\pi^{n-p+\frac{1}{2}}}{(n-p+\frac{1}{2})!} \sum_{c=0}^{p} \frac{k_{2c}(P)\{(t+r)^{2n-2p+2c+1} - (t-r)^{2n-2p+2c+1}\}}{(2n-2p+3)\cdots(2n-2p+2c+1)}$$

$$= \frac{\pi^{n-p+\frac{1}{2}}}{(n-p+\frac{1}{2})!} \sum_{c=0}^{p} \frac{k_{2c}(P)}{(2n-2p+3)\cdots(2n-2p+2c+1)}$$
$$\cdot \sum_{a=0}^{n-p+c} \binom{2n-2p+2c+1}{2a+1} 2r^{2a+1} t^{2(n-p+c-a)}.$$

Letting $b = c - a + n - p$ and changing the order of summation, we obtain

$$V_P^{\mathbb{R}^{2n+1}}(t+r) - V_P^{\mathbb{R}^{2n+1}}(t-r) \tag{5.4}$$
$$= \frac{2\pi^{n-p+\frac{1}{2}}}{(n-p+\frac{1}{2})!} \sum_{a=0}^{n} \frac{k_{2(b+a-n+p)}(P) r^{2a+1}}{(2n-2p+3)\cdots(2a+2b+1)} \sum_{b=0}^{n-a} \binom{2a+2b+1}{2a+1} t^{2b}.$$

Now (5.1) follows from (5.2)–(5.4) when we equate the coefficients of like powers of r. $\square$

A special case is noteworthy.

Corollary 5.2. *Under the assumptions of Theorem* 5.1 *we have*

$$k_{2n}(P_t) = 2(2\pi)^{n-p} k_{2p}(P). \tag{5.5}$$

Proof. When $a = n$ the sum on the right-hand side of (5.1) collapses to one term. Thus (5.5) follows from

$$\frac{k_{2n}(P_t)}{1\cdot 3\cdots(2n+1)} = \frac{\pi^{n-p+\frac{1}{2}} k_{2p}(P)}{(n-p+\frac{1}{2})!\,(2n-2p+3)\cdots(2n+1)}$$
$$= \frac{2(2\pi)^{n-p} k_{2p}(P)}{1\cdot 3\cdots(2n+1)}.$$

$\square$

5.2 The Euler Characteristic

The most familiar definition of the **Euler characteristic** of a compact manifold is the alternating sum of Betti numbers:

$$\chi(P) = \sum_{a=0}^{\dim P} (-1)^a b_a(P).$$

Here the a^{th} **Betti number** $b_a(P)$ is the dimension of the cohomology group $H^a(P, \mathbb{R})$, or equivalently the dimension of the space of harmonic a-forms. However, often it is useful in differential geometry to use a more elementary definition, namely

$$\chi(P) = \sum_{a=0}^{\dim P} (-1)^a \alpha_a(P),$$

where $\alpha_a(P)$ is the number of a-simplexes of P with respect to some triangulation. Then $\chi(P)$ (but not the $\alpha_a(P)$'s) is independent of the choice of triangulation. Counting arguments can be used to prove that

$$\chi(E) = \chi(F)\,\chi(B), \tag{5.6}$$

for any differentiable fibration of $E \longrightarrow B$ with fiber F, where E, F and B are compact [Hu, page 277]. In particular,

$$\chi(F \times B) = \chi(F)\,\chi(B), \tag{5.7}$$

$$\chi(\tilde{P}) = p\,\chi(P), \tag{5.8}$$

where $\tilde{P} \longrightarrow P$ is a p-fold covering space.

We shall need the following related fact:

Lemma 5.3. *Let $P^p \subset \mathbb{R}^n$ be a compact submanifold. Then*

$$\chi(P_r) = \chi(P)\,\chi\big(S^{n-p-1}(1)\big). \tag{5.9}$$

Proof. It is not hard to see that P_r fibers topologically over P with fiber $S^{n-p-1}(r)$. Hence (5.9) follows from (5.6) and the obvious fact that

$$\chi\big(S^{n-p-1}(1)\big) = \chi\big(S^{n-p-1}(r)\big)$$

for any $r > 0$.[1] $\square$

Corollary 5.4. *Let $P \subset \mathbb{R}^{2n+1}$ be a compact submanifold of dimension $2p$. Then the Euler characteristics of P and a parallel tubular hypersurface P_t are related by*

$$\chi(P_t) = 2\chi(P).$$

Proof. In this case the fiber is a sphere of even dimension $2n - 2p$. $\square$

[1] Here and infrequently elsewhere we use r to denote the radius of a sphere instead of the sectional curvature.

5.3 The Pfaffian

When the dimension of P is even, say $2p$, a particularly important coefficient is the top one, $k_{2p}(P)$. It is the integral of an interesting curvature function, often called the Gauss-Bonnet integrand. It is best understood in terms of a certain function of matrices, the Pfaffian,[2] which can be described as a square root of the determinant.

Definition. *Let A be any $2m \times 2m$ antisymmetric matrix; then the* **Pfaffian** $\mathrm{Pf}(A)$ *is defined by*

$$\mathrm{Pf}(A) = \frac{1}{2^m m!} \sum_{\rho \in \mathfrak{S}_{2m}} \varepsilon_\rho A_{\rho_1 \rho_2} \cdots A_{\rho_{2m-1} \rho_{2m}}. \tag{5.10}$$

Many terms on the right-hand side of (5.10) coincide with one another. When we combine them, the formula for the Pfaffian reduces to

$$\mathrm{Pf}(A) = {\sum}' \varepsilon_\rho A_{\rho_1 \rho_2} \cdots A_{\rho_{2m-1} \rho_{2m}}, \tag{5.11}$$

where now the sum $\sum'$ is over all permutations $\rho \in \mathfrak{S}_{2m}$ such that

$$\rho_1 < \rho_3 < \cdots < \rho_{2m-1} \quad \text{and} \quad \rho_{2i-1} < \rho_{2i} \quad (1 \le i \le m).$$

For example, $\mathrm{Pf}\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a$ and

$$\mathrm{Pf}\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

One useful method for computing higher order Pfaffians is as follows. Let $A_{i_1 \dots i_{2k}}$ denote the Pfaffian of the submatrix $(A_{i_p i_q})_{1 \le p,q \le 2k}$. From (5.11) it follows by induction that

$$A_{i_1 \dots i_{2k}} = \sum_{p=2}^{2k} (-1)^p A_{i_1 i_p} A_{i_2 \dots \hat{i}_p \dots i_{2k}}.$$

For the computation of the Pfaffian of a matrix that is antisymmetric and of the form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

see Lemma 6.6 of Chapter 6.

[2] Johann Friedrich Pfaff (1765–1825). German mathematician, professor of mathematics at Helmstedt from 1788 to 1810 when he was appointed to the chair of mathematics at Halle. He worked in the areas of partial differential equations, special functions and the theory of series. His most important work on Pfaffian forms was published in 1815, but its importance was not recognized until 1827 when Jacobi published a paper on Pfaff's method. Pfaff taught Gauss, who lived in his house in 1798. He recommended Gauss's doctoral thesis.

The most important fact about the Pfaffian is that it is a square root of the determinant:

$$\mathrm{Pf}(A)^2 = \det(A). \tag{5.12}$$

Another important property is:

$$\mathrm{Pf}(CD^tC) = \mathrm{Pf}(D)\det(C), \tag{5.13}$$

where ${}^tD = -D$ and C is arbitrary. (Here ${}^tD$ denotes the transpose of D.) For a straightforward inductive proof of (5.12) see [Chern4]. Other good references for the Pfaffian are [MS, pages 309–310], and [Spivak, volume 5, pages 416–417], where (5.13) is proved.

So far we have been dealing with the Pfaffian of a $2m \times 2m$ matrix A with ${}^tA = -A$, but we have said nothing else about the nature of the entries of A. In fact, it is clear that $\mathrm{Pf}(A)$, like $\det(A)$, makes sense not only for real and complex A, but also for A with entries from a commutative ring, say the ring $\Lambda^{\mathrm{e}}(M)$ of differential forms of even degree on a manifold M. We now exploit this fact.

Let M be a $2n$-dimensional Riemannian manifold and $\{E_1, \dots, E_{2n}\}$ a local orthonormal frame. Then the **curvature forms** Ω_{ij} $(1 \leq i, j \leq 2n)$ of M with respect to this frame are given by

$$\Omega_{ij}(X,Y) = \langle R_{XY} E_i, E_j \rangle.$$

So the matrix of curvature forms $\Omega = (\Omega_{ij})$ is an antisymmetric matrix of 2-forms, and $\mathrm{Pf}(\Omega)$ is a well-defined $2n$-form.

Definition. *Let M be a Riemannian manifold of dimension $2n$, and let Ω be the matrix of curvature forms relative to a local orthonormal frame $\{E_1, \dots, E_{2n}\}$. The* **Euler form** *of M (relative to $\{E_1, \dots, E_{2n}\}$) is*

$$\frac{1}{(2\pi)^n}\,\mathrm{Pf}(\Omega).$$

The factor $(2\pi)^{-n}$ is included so that the Euler form exactly coincides with the Chern form of highest degree that we shall define in Chapter 6, and also to make the Generalized Gauss-Bonnet Theorem look a little nicer. The **Gauss-Bonnet integrand** is by definition the Euler form.

Lemma 5.5. *Let M be an oriented Riemannian manifold of dimension $2p$ with Riemannian volume element ω. Then the definition of the Euler form χ does not depend on the choice of local orthonormal frame field among those local orthonormal frames that define the given orientation of M. Furthermore, we have*

$$R^p(E_1, \dots, E_{2p})(E_1, \dots, E_{2p}) = p!\,\langle \mathrm{Pf}(\Omega), \omega \rangle. \tag{5.14}$$

Proof. It follows from (4.3), page 55, and induction that

$$R^p(X_1,\ldots,X_{2p}) = \frac{1}{2^p}\sum_{\rho\in\mathfrak{S}_{2p}} \varepsilon_\rho R(X_{\rho_1},X_{\rho_2})\wedge\ldots\wedge R(X_{\rho_{2p-1}},X_{\rho_{2p}}) \tag{5.15}$$

for $X_1,\ldots,X_{2p}\in\mathfrak{X}(M)$. Then (5.14) is obvious from (5.15) and the definition of the Pfaffian.

Furthermore, let $\{F_1,\ldots,F_{2p}\}$ be any other local orthonormal frame field. We can write $F_i = \sum a_{ij}E_j$, where (a_{ij}) is some orthogonal matrix. Then

$$R^p(F_1,\ldots,F_{2p}) = \varepsilon R^p(E_1,\ldots,E_{2p}),$$

where $\varepsilon = \det(a_{ij}) = \pm 1$, so that

$$R^p(F_1,\ldots,F_{2p})(F_1,\ldots,F_{2p}) = R^p(E_1,\ldots,E_{2p})(E_1,\ldots,E_{2p}).$$

Thus from (5.14) it follows that $\langle \mathrm{Pf}(\Omega),\omega\rangle$ depends neither upon the choice of local frame nor upon the orientation. (When the orientation of M is changed, ω is replaced by $-\omega$.) On the other hand, χ does depend on the orientation, but is otherwise independent of the choice of the local orthonormal frame. $\square$

There is a close relation between the Euler form and the top coefficient $k_{2p}(P)$ in Weyl's Tube Formula.

Corollary 5.6. *Let P be a compact Riemannian manifold of even dimension $2p$, and denote by Ω^P its matrix of curvature forms (with respect to any local orthonormal frame). Then*

$$k_{2p}(P) = \int_P \mathrm{Pf}(\Omega^P). \tag{5.16}$$

Proof. First, we assume that P is orientable. For any curvature tensor R we have

$$C^{2p}(R^p) = (2p)!\,R^p(E_1,\ldots,E_{2p})(E_1,\ldots,E_{2p}). \tag{5.17}$$

In particular, for the curvature tensor R^P of P we have from (4.4), page 56, and (5.17) that

$$k_{2p}(P) = \frac{1}{p!\,(2p)!}\int_P C^{2p}((R^P)^p)\,dP = \int_P \langle \mathrm{Pf}(\Omega^P),\omega\rangle\,\omega = \int_P \mathrm{Pf}(\Omega^P),$$

proving (5.16) in the case that P is orientable.

Let $-P$ denote P with the opposite orientation. Then

$$C^{2p}((R^{-P})^p) = C^{2p}((R^P)^p),$$

because P and $-P$ have the same curvature tensor, and contraction is independent of the orientation. Moreover, by (5.14) we have

$$\int_P \mathrm{Pf}(\Omega^P) = -\int_P \mathrm{Pf}(\Omega^{-P}) = \int_{-P} \mathrm{Pf}(\Omega^{-P}).$$

If P is not orientable, it can be split up into orientable pieces. Since (5.16) holds for each piece and neither side depends on the orientation, it follows from the additivity of the integral that (5.16) must hold for nonorientable as well as orientable manifolds. $\square$

5.4 The Gauss-Bonnet Theorem for Hypersurfaces in $\mathbb{R}^{2n+1}$

We first recall Hopf's[3] formula [Hopf1], [Hopf2] for the Euler characteristic of a compact hypersurface of $\mathbb{R}^{2n+1}$.

Theorem 5.7. (Hopf's Theorem.) *Let $P^{2n} \subset \mathbb{R}^{2n+1}$ be a compact embedded hypersurface. Then*

$$(2\pi)^n \chi(P) = 1 \cdot 3 \cdots (2n-1) \int_P \kappa_1 \cdots \kappa_{2n} \, dP, \tag{5.18}$$

where the κ_i's are the principal curvatures of P.

We refer to [Spivak, volume 5, page 386], for a proof of Hopf's Theorem. Notice, however, that for a sphere $S^{2n}(r)$ of radius r we have $\chi\big(S^{2n}(r)\big) = 2$ and

$$\begin{aligned}
\int_{S^{2n}(r)} \kappa_1 \cdots \kappa_{2n} \, dP &= \frac{1}{r^{2n}} \mathrm{volume}\big(S^{2n}(r)\big) = \frac{1}{r^{2n}} \frac{d}{dr} \left(\frac{(\pi r^2)^{n+\frac{1}{2}}}{(n+\frac{1}{2})!} \right) \\
&= \frac{r^{-2n} (\pi r^2)^{n-\frac{1}{2}} (2\pi r)}{(n-\frac{1}{2})!} \\
&= \frac{2\pi^{n+\frac{1}{2}}}{(n-\frac{1}{2})(n-\frac{3}{2}) \cdots (\frac{1}{2})(-\frac{1}{2})!} = \frac{2(2\pi)^n}{1 \cdot 3 \cdots (2n-1)}.
\end{aligned}$$

Hence (5.18) gives the right answer for even dimensional spheres of any radius.

---

[3] Heinz Hopf (1894–1971). Swiss mathematician, professor at the Eidgenössische Technische Hochschule in Zürich. The greater part of his work was in algebraic topology, motivated by an exceptional geometric intuition. In 1931 Hopf studied homotopy classes of maps between the spheres S^3 to S^2 and defined what is now known as the Hopf invariant.

Let $f\colon S^{2n+1} \longrightarrow S^n$ be a differentiable mapping between spheres, and let Ω be an n-form such that $[\Omega]$ is a generator for the cohomology group $H^n(S^n, \mathbb{Z})$. Then the form $f^*(\Omega)$ is a closed form of degree n on S^{2n+1}. It is exact because the group $H^n(S^{2n+1}, \mathbb{Z})$ is trivial; thus $f^*(\Omega) = d\vartheta$ for some form ϑ of degree $n-1$. In 1937 J.H.C. Whitehead showed that the Hopf invariant is given by

$$H(f) = \int_{S^{2n-1}} \vartheta \wedge d\vartheta.$$

Hopf is responsible for other important concepts in modern topology such as Hopf fibration, Hopf group and Hopf algebra.

Note also that although each principal curvature κ_i changes to $-\kappa_i$ when the orientation is changed, the product $\kappa_1 \cdots \kappa_{2n}$ is independent of orientation. So, the left-hand side does not depend on the orientation; obviously the Euler characteristic is independent of orientation.

For the proof of the next theorem we need some special 1-forms.

Definition. *Let $\{E_1, \ldots, E_n\}$ be a local orthonormal frame field on an n-dimensional Riemannian manifold M. Then the* **dual 1-forms** *of M with respect to $\{E_1, \ldots, E_n\}$ are the differential forms θ_i defined by*

$$\theta_i(X) = \langle X, E_i \rangle$$

for $i = 1, \ldots, n$ and $X \in \mathfrak{X}(M)$.

Theorem 5.8. (The Gauss-Bonnet Theorem for hypersurfaces in $\mathbb{R}^{2n+1}$) *Let $P^{2n} \subset \mathbb{R}^{2n+1}$ be a compact hypersurface. Then*

$$(2\pi)^n \chi(P) = \int_P \operatorname{Pf}(\Omega^P)\, dP = k_{2n}(P). \tag{5.19}$$

Proof. Let $\{E_1, \ldots, E_{2n}\}$ be a local orthonormal frame defined on an open orientable subset $U \subset P$, and let $\{\theta_1, \ldots, \theta_{2n}\}$ be the dual 1-forms. Since U is an orientable hypersurface, it has a globally defined normal vector field N. Let S be the corresponding shape operator. (So, just as in Chapter 2, $SX = -\nabla_N X$.) We assume that each E_i is a principal curvature vector field, that is, $SE_i = \kappa_i E_i$. It is easy to check that the curvature forms Ω_{ij} of P with respect to this frame field are given by

$$\Omega_{ij} = \kappa_i \kappa_j\, \theta_i \wedge \theta_j. \tag{5.20}$$

(The computation using the Gauss equation (4.19) is:

$$\begin{aligned}\Omega_{ij}(X,Y) &= \langle R_{E_i E_j} X, Y\rangle = \langle SE_i, X\rangle\langle SE_j, Y\rangle - \langle SE_i, Y\rangle\langle SE_j, X\rangle \\ &= \kappa_i\kappa_j \Big(\theta_i(X)\theta_j(Y) - \theta_i(Y)\theta_j(X)\Big) = \kappa_i\kappa_j\,(\theta_i \wedge \theta_j)(X,Y).)\end{aligned}$$

Using (5.10) and (5.20) it is easy to compute the Euler form:

$$\operatorname{Pf}(\Omega) = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \varepsilon_\sigma\, \Omega_{\sigma_1\sigma_2} \wedge \ldots \wedge \Omega_{\sigma_{2n-1}\sigma_{2n}} \tag{5.21}$$

$$= \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \varepsilon_\sigma\, \kappa_{\sigma_1} \cdots \kappa_{\sigma_{2n}} \theta_{\sigma_1} \wedge \theta_{\sigma_2} \wedge \ldots \wedge \theta_{\sigma_{2n-1}} \wedge \theta_{\sigma_{2n}}$$

$$= \frac{\kappa_1 \cdots \kappa_{2n}}{2^n n!} (2n)!\, \theta_1 \wedge \ldots \wedge \theta_{2n}$$

$$= 1 \cdot 3 \cdots (2n-1) \kappa_1 \cdots \kappa_{2n}\, \omega,$$

where ω is the Riemannian volume element of P compatible with the given orientation. Since the product $\kappa_1 \cdots \kappa_{2n}$ is independent of the choice of orientation, we get (5.19) when we integrate (5.21) over P. □

5.5 The Tube Proof of the Generalized Gauss-Bonnet Theorem

We state without proof the following fact (see [Nash]):

Nash Embedding Theorem. *Let P be a compact Riemannian manifold. Then for some n there is an isometric embedding of P in $\mathbb{R}^n$.*

Now we have all the facts at hand to give the proof of Allendoerfer [Al1] and Fenchel [Fl2] of the Generalized Gauss-Bonnet Theorem. The original proof was for a submanifold of some Euclidean space, but the Nash Embedding Theorem can be used to eliminate this hypothesis.

Theorem 5.9. (The Generalized Gauss-Bonnet Theorem.) *Let P be a $2p$-dimensional compact Riemannian manifold. Then the Euler characteristic is expressed in terms of the curvature of P by the formula*

$$(2\pi)^p \chi(P) = k_{2p}(P) = \int_P \mathrm{Pf}(\Omega^P)\, dP. \tag{5.22}$$

Proof. By the Nash Embedding Theorem it is possible to embed P isometrically in some Euclidean space. We can assume (by increasing the codimension by 1, if necessary) that this Euclidean space is an odd dimensional space $\mathbb{R}^{2n+1}$. Choose a $t > 0$ so that

$$P_t = \{\, m \in \mathbb{R}^{2n+1} \mid \text{distance}(m, P) = t \,\}$$

is a well-defined hypersurface. This is possible because $\exp_\nu$ is a diffeomorphism in a neighborhood of P. By Corollary 5.2 and the Gauss-Bonnet Theorem for hypersurfaces, we have

$$k_{2p}(P) = \frac{1}{2}(2\pi)^{-n+p} k_{2n}(P_t) = \frac{1}{2}(2\pi)^{-n+p}(2\pi)^n \chi(P_t).$$

Thus by Corollary 5.4 the Euler characteristic of P is given by the formula

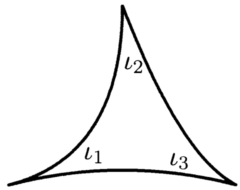
$$k_{2p}(P) = (2\pi)^p \chi(P).$$

The second equality of (5.22) follows from (5.16). □

5.6 The History of the Gauss-Bonnet Theorem

The original theorem of Gauss concerns a triangle T whose sides are geodesics on a surface M. See Dombrowski [Dom] for a translation of Gauss' *Disquisitiones Generales Circa Superficies Curvas*, a guide to commentaries on this work and many enlightening remarks on how to read it from a modern point of view. See also [Gauss1] and [Gauss2]. Gauss' theorem states that

$$\int_T K\,dM = \iota_1 + \iota_2 + \iota_3 - \pi,$$



where the ι_i are the angles of the triangle T. By piecing together triangles on a compact surface Bonnet[4] [Bonnet] used Gauss' result to show that

$$2\pi\chi(M) = \int_M K\,dM.$$

This is the most familiar form of the Gauss-Bonnet Theorem for surfaces. Modern proofs, following Darboux [Da], are usually based on Stokes' Theorem. For discussions of the early history of the Gauss-Bonnet Theorem see [Struik, pages 153–156], [Bühler, pages 103–106] and [Dom].

For many years it was unclear how to generalize the 2-dimensional Gauss-Bonnet Theorem to higher dimensions. It is reasonable to exclude odd dimensional compact manifolds, since they all have Euler characteristic zero. The next step came in 1925 when H. Hopf in his papers [Hopf1] and [Hopf2] gave a formula for the Euler characteristic for a compact orientable hypersurface P^{2n} of $\mathbb{R}^{2n+1}$ (Theorem 5.8). His proof makes use of the Gauss map from P^{2n} to the unit sphere of $\mathbb{R}^{2n+1}$.

But hypersurfaces of Euclidean space are very special manifolds. (For instance, for $n \geq 2$ it is not possible to embed complex projective space $\mathbb{C}P^n$ as a hypersurface in Euclidean space $\mathbb{R}^{2n+1}$.) However, the Nash Embedding Theorem states that it is possible isometrically to embed every compact Riemannian manifold in some Euclidean space. (This theorem is much more difficult than the Whitney Embedding Theorem, which says that every differentiable manifold M^n can be differentiably embedded in $\mathbb{R}^{2n+1}$ as a closed subset (see [BJ, page 71]). Also, the dimension of the embedding space for an isometric immersion is much

[4] Pierre Ossian Bonnet (1819–1892). French mathematician, who made many important contributions to surface theory, including the Gauss-Bonnet Theorem. Bonnet was director of studies at the École Polytechnique, professor of astronomy in the faculty of sciences at the University of Paris, and member of the board of longitudes.

larger than the dimension of the embedding space for a differentiable immersion.) On the other hand, Allendoerfer [All] and Fenchel [Fl2] proved the Gauss-Bonnet Theorem for a Riemannian manifold M that was assumed to be in some Euclidean space. As we have seen, this proof (which is essentially that of Theorem 5.9) depends on Weyl's Tube Formula. In view of the Nash Embedding Theorem, this is sufficient to establish the theorem in general.

However, the Generalized Gauss-Bonnet Theorem was actually proved before the Nash Embedding Theorem; the first proof was given in [AW] by Allendoerfer and Weil[5] in 1943. Their proof relies on the Allendoerfer-Fenchel proof of the Generalized Gauss-Bonnet Theorem for submanifolds of Euclidean space together with the Cartan-Janet-Burstin Theorem ([Cal], [Burstin]) The Cartan-Janet-Burstin Theorem is a weak version of the Nash Embedding Theorem; it states that any Riemannian manifold has a *local isometric embedding* into some Euclidean space. So Allendoerfer and Weil embed an even dimensional Riemannian manifold one piece at a time into Euclidean space, and then they give a rather complicated argument to fit the pieces together. This argument, which is of independent interest, involves a generalization to higher dimensions of the Gauss-Bonnet Theorem for surfaces with boundary.

Chern's intrinsic proof [Chern1] of the Generalized Gauss-Bonnet Theorem goes as follows. Let P be a $2q$-dimensional compact orientable Riemannian manifold and let $\pi\colon S(P) \longrightarrow P$ be the unit sphere bundle. First, observe that when the Euler form $(2\pi)^{-q}\,\mathrm{Pf}(\Omega^P)$ is pulled back to $S(P)$ it is exact. Thus there exists a $(2q-1)$-form Φ on $S(P)$ such that

$$\pi^*\left(\frac{1}{(2\pi)^q}\,\mathrm{Pf}(\Omega^P)\right) = d\Phi.$$

(See also [Fla].)

[5] André Weil (1906–1998). French mathematician, one of the most respected mathematicians of the 20^{th} century. Weil was hailed for bringing together number theory and algebraic geometry, laying the groundwork for such areas as the theory of modular forms, automorphic functions and automorphic representations. His work has found application in elementary particle physics and in the development of modern mathematical cryptography. Weil is best known for two things: his fundamental discoveries in number theory, and his membership of the secretive group known as Bourbaki, which redefined the foundations of modern pure mathematics.

After writing a thesis under the direction of Jacques Hadamard he broke the European academic mold by beginning his teaching career in 1930 at India's Aligarh Muslim University. Later in the 1930s, he returned to France, teaching at the University of Strasbourg. In 1939 Weil was called for military service. He fled to Finland, explaining that while he could be of some use doing mathematics, "as a soldier I would be entirely useless". He was saved from execution as a spy by the lucky intervention of Rolf Nevanlinna. He was sent back to France and spent six months in a military prison before agreeing to join the French army. After the fall of France, he made his way to the United States, holding several teaching posts before joining the Institute for Advanced Study in 1958.

Weil was the brother of the well-known religious mystic and author Simone Weil. In 1994 Weil received the Kyoto Prize.

Let Y be a C^∞ vector field on P with isolated zeros. Then $X = \|Y\|^{-1}Y$ has isolated singularities and can be viewed as a section

$$X\colon P - \{\text{singularities}\} \longrightarrow S(P).$$

Moreover, the boundary of image $X(P - \{\text{singularities}\})$ is a $(2q-1)$-dimensional cocycle of $S(P)$. Therefore, by Stokes' Theorem

$$\begin{aligned}\frac{1}{(2\pi)^q}\int_P \mathrm{Pf}(\Omega^P) &= \frac{1}{(2\pi)^q}\int_{P-\{\text{singularities}\}} \mathrm{Pf}(\Omega^P)\\ &= \int_{P-\{\text{singularities}\}} X^*(d\Phi)\\ &= \int_{X(P-\{\text{singularities}\})} d\Phi\\ &= \int_{\partial X(P-\{\text{singularities}\})} \Phi.\end{aligned}$$

This last expression is the sum of the indices of the vector field X, which by a well-known result from algebraic topology equals the Euler characteristic $\chi(P)$. Thus we get (5.22).

In [Chern2] Chern used the same techniques to give a proof of the Generalized Gauss-Bonnet Theorem for a Riemannian manifold with smooth boundary. The Generalized Gauss-Bonnet Theorem for a Riemannian manifold with nonsmooth boundary also holds, but then the more complicated proof of Allendoerfer and Weil [AW] must be used.

5.7 Problems

5.1 Show that the volume of a tube about a compact surface P in $\mathbb{R}^n$ is given by the formula

$$V_P^{\mathbb{R}^n}(r) = \frac{(\pi r^2)^{\frac{1}{2}n-1}}{\left(\frac{1}{2}n-1\right)!}\left\{\mathrm{volume}(P) + \frac{2\pi\chi(P)r^2}{n}\right\}.$$

5.2 Show that the volume of a tube about a compact surface P in a space $\mathbb{K}^n(\lambda)$ of constant curvature λ is given by the formula

$$\begin{aligned}\frac{d}{dr}V_P^{\mathbb{K}^n(\lambda)}(r) &= \frac{2\pi^{\frac{1}{2}n-1}}{\Gamma\left(\frac{1}{2}n-1\right)}\left(\frac{\sin(r\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-3}\\ &\cdot\left\{\mathrm{volume}(P)\left(1-\frac{n-1}{n-2}\sin^2(r\sqrt{\lambda})\right) + \frac{2\pi\chi(P)\sin^2(r\sqrt{\lambda})}{(n-2)\lambda}\right\}.\end{aligned}$$

Chapter 6

Chern Forms and Chern Numbers

We interrupt our study of tubes in order to present some basic information about complex manifolds that we shall need in Chapter 7 when we prove the Complex and Projective Weyl Tube Formulas. In Section 6.1 we start by recalling some basic facts about Kähler manifolds. Then we define the Chern forms of the tangent bundle of a Kähler manifold in terms of curvature and discuss the basic properties of these forms. Section 6.2 is devoted to Kähler manifolds with constant holomorphic sectional curvature. In Section 6.3 we recall some facts about locally symmetric spaces that we shall eventually need. More can be said about the volume of a submanifold P of a Riemannian manifold M, provided the second fundamental form of P is not too complicated when compared with the curvature operator of M. Therefore, in Section 6.3 we define the notion of compatible submanifold and study the basic properties of such submanifolds. We derive Study's formula [Study] for the volume $V_m^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}(r)$ of a geodesic ball of radius r in a space $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ of constant holomorphic sectional curvature in Section 6.4. Complex projective space $\mathbb{C}P^n(\lambda)$ is discussed in Section 6.5, where we use Study's formula for $V_m^{\mathbb{C}P^n(\lambda)}(r)$ to find the volume of $\mathbb{C}P^n(\lambda)$. Then in Section 6.6, we compute the total Chern form of $\mathbb{C}P^n(\lambda)$ directly from its definition in terms of curvature. A brief treatment of Wirtinger's Inequality is given in Section 6.7. In Section 6.8 we write down representatives of the generators of the integral cohomology of $\mathbb{C}P^n(\lambda)$ that are useful. In Section 6.9 we discuss Chern numbers, in particular the Chern numbers of complex projective space $\mathbb{C}P^n(\lambda)$. The chapter concludes with an explicit computation in Section 6.10 of the Chern numbers of a complex hypersurface of $\mathbb{C}P^n(\lambda)$ in terms of the degree of the hypersurface.

6.1 The Chern Forms of a Kähler Manifold

A Kähler[1] manifold is a special kind of Riemannian manifold that has many interesting properties. We shall see in Chapter 7 that Weyl's Tube Formula simplifies dramatically for a tube about a Kähler submanifold of complex Euclidean space. For a compact Kähler manifold M (or more generally for a compact almost complex manifold) there is a sequence of integers, the Chern numbers, that generalize the Euler characteristic. Each can be defined as an integral over M of a certain differential form, a Chern form. In this section we define the Chern forms in terms of the curvature tensor and establish some formulas involving them that we shall need when we study tubes about Kähler submanifolds.

To explain the notion of Kähler manifold, we need some auxiliary definitions. An **almost complex structure** on a differentiable manifold M is a tensor field J of type $(1,1)$ such that $J^2 = -I$. (Thus J is a field of linear transformations $m \longmapsto J_m$, where each $J_m : M_m \longrightarrow M_m$ is a linear map such that $J_m^2 = -I_m$, where I_m is the identity map on M_m.)

Lemma 6.1. *An almost complex manifold M has even dimension.*

Proof. **(Following [Calabi1].)** Let $n = \dim M$. Then for any $m \in M$ we have

$$(-1)^n = \det(-I_m) = \det(J_m^2) = (\det(J_m))^2. \qquad \square$$

Suppose that M is simultaneously a Riemannian manifold and an almost complex manifold. It is reasonable to require compatibility between the almost complex structure J and the metric tensor $\langle\ ,\ \rangle$, and the most reasonable compatibility condition is to require that J be an isometry: $\langle JX, JY\rangle = \langle X, Y\rangle$ for $X, Y \in \mathfrak{X}(M)$. If this is the case, we say that $\langle\ ,\ \rangle$ is an **almost Hermitian metric**, and that M is an almost Hermitian manifold. The **Kähler form** F of an almost Hermitian manifold M is the 2-form defined by $F(X,Y) = \langle JX, Y\rangle$ for $X, Y \in \mathfrak{X}(M)$.

Lemma 6.2. *Any almost complex manifold is orientable and can be made into an almost Hermitian manifold.*

Proof. Since we are assuming that M is paracompact, it has a Riemannian metric $(\ ,\ )$. We define a new metric $\langle\ ,\ \rangle$ on M by $\big\langle X, Y\big\rangle = (X,Y) + (JX, JY)$ for $X, Y \in \mathfrak{X}(M)$. Then it is easy to check that $\langle\ ,\ \rangle$ is an almost Hermitian metric

[1] Erich Kähler (1906–). German mathematician. Kähler studied mathematics, astronomy and physics at the University of Leipzig. He taught at the Universities of Königsberg, Hamburg, Leipzig, and the Technische Universität, Berlin. In his **Geometrica arithmetica** he synthesized arithmetic, algebraic geometry and function theory.

Kähler manifolds seem to have been defined for the first time by Kähler in 1933 (*Hamb. Abh.* **9** (1933)), but as Weil points out in [Weil] the importance of Kähler manifolds only became apparent in Chapter V of Hodge's great work [Hodge]. Weil's important book [Weil] systematically described the cohomology of a compact Kähler manifold. This was complemented by the paper [DGMS] in which it is shown that the minimal model of a compact Kähler manifold is formal.

on M. Furthermore, the Kähler form F of this metric has the property that F^n is everywhere nonzero on M, where $\dim M = 2n$. This implies that M is orientable. $\square$

(In Lemma 6.28 we give more precise information about F^n.)

At this point it will be convenient to alter the notation slightly. We shall write R_{vwxy} for the value of the curvature tensor field on tangent vectors v, w, x, y at some point $m \in M$. (Since R is a tensor field, we have $(R_{VWXY})_m = R_{vwxy}$ for any vector fields V, W, X, Y such that $V_m = v$, and so forth.) Also, sometimes we put $Jx = x^*$.

Furthermore, we shall need various tensor fields that assume complex values (for example, the complex curvature forms). It is possible to use complex functions and tangent vectors to make these definitions, but this is not necessary. At any rate, most formulas for Riemannian manifolds hold with complex arguments when they are complexified in the canonical way. Obvious exceptions to this statement are inequalities involving positive definiteness of the metric.

Let M be an almost Hermitian manifold and let $m \in M$. Frequently, it will be convenient to use a special orthonormal basis of the tangent space M_m. We call a real orthonormal basis of the form $\{e_1, Je_1, \ldots, e_n, Je_n\}$ a **holomorphic orthonormal frame**, and we shall usually use the notation $\{1, 1^*, \ldots, n, n^*\}$. (So, $Ji = i^*$ and $Ji^* = -i$.) When a holomorphic orthonormal frame is used to form some contraction, the contraction can be written, if necessary, as a sum from 1 to n instead of from 1 to $2n$.

Although there are many interesting almost Hermitian manifolds, Kähler manifolds form by far the most interesting subclass. For $X \in \mathfrak{X}(M)$ write $\nabla_X(J) = \nabla_X J - J\nabla_X$.

Definition. *A* **Kähler manifold** *is an almost Hermitian manifold M for which the almost complex structure J of M is parallel, that is, $\nabla_X(J)Y = 0$ for $X, Y \in \mathfrak{X}(M)$.*

The curvature tensor of a Kähler manifold M satisfies identities in addition to (2.16)–(2.19):

Lemma 6.3. *Let M be a Kähler manifold. Then for $X, Y \in \mathfrak{X}(M)$ we have*

$$R_{JXJY} = R_{XY}, \tag{6.1}$$

$$\rho(JX, JY) = \rho(X, Y). \tag{6.2}$$

Proof. Equation (6.1) (called the **Kähler identity**) is an easy consequence of the definition of R_{XY} and the fact that $\nabla_X(J)Y = 0$.

Let $\{E_1, \ldots, E_{2n}\}$ be any (real) local orthonormal frame field on a Kähler manifold M. Then $\{JE_1, \ldots, JE_{2n}\}$ is also a local orthonormal frame field. From this trivial observation and (6.1), it follows that the Ricci curvature ρ of M satisfies (6.2). $\square$

Thus using the first Bianchi identity (2.19) together with (6.2), we see that for a Kähler manifold M the Ricci and scalar curvatures can be expressed in terms of any holomorphic orthonormal frame $\{1, 1^*, \ldots, n, n^*\}$ as

$$\rho(x,y) = \sum_{i=1}^{n} R_{xy^*ii^*}, \qquad \tau = \sum_{i,j=1}^{n} R_{ii^*jj^*} \tag{6.3}$$

for tangent vectors x, y to M.

Throughout the rest of this chapter, unless stated otherwise, any almost complex manifold will be assumed to be Kählerian.

In order to define Chern forms we shall need complex variants of the real curvature forms we defined in Chapter 5. Let $m \in M$ and fix a holomorphic orthonormal frame $\{1, 1^*, \ldots, n, n^*\}$.

Definition. *The* **complex curvature forms** Ξ_{ij} $(1 \le i, j \le n)$ *of a Kähler manifold M relative to a holomorphic orthonormal frame $\{1, 1^*, \ldots, n, n^*\}$ are given by*

$$\Xi_{ij} = \Omega_{ij} - \sqrt{-1}\,\Omega_{ij^*}.$$

Note that while the matrix (Ω_{ij}) of real curvature forms is an antisymmetric matrix, the matrix (Ξ_{ij}) of complex curvature forms is anti-Hermitian by Lemma 6.3. The complex curvature forms have been introduced so that we can give a simple definition of Chern form, and later, Chern class and Chern number.

Definition. *Let M be a Kähler manifold. Write*

$$\det_{1 \le i,j \le n}\left(\delta_{ij} - \frac{1}{2\pi\sqrt{-1}}\,\Xi_{ij}\right) = 1 + \gamma_1 + \cdots + \gamma_n = \gamma. \tag{6.4}$$

Then γ is called the **total Chern form** *of M. Its component γ_i of degree $2i$ is called the i^{th}* **Chern form**.

Before proceeding further we observe some elementary facts about Chern forms.

Lemma 6.4. *For $i = 1, \ldots, n$ the i^{th} Chern form γ_i is a real differential form.*

Proof. It suffices to show that the total Chern form γ equals its complex conjugate $\overline{\gamma}$. From the fact that $\overline{\Xi}_{ij} = -\Xi_{ji}$, we get

$$\begin{aligned}\overline{\gamma} &= \overline{\det_{1 \le i,j \le n}\left(\delta_{ij} - \frac{1}{2\pi\sqrt{-1}}\,\Xi_{ij}\right)} \\ &= \det_{1 \le i,j \le n}\left(\delta_{ij} - \frac{1}{2\pi\sqrt{-1}}\,\Xi_{ji}\right) = \gamma. \qquad \square\end{aligned}$$

Two Riemannian metrics $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle'$ are called **homothetic**, provided there exists a constant μ such that $\langle\ ,\ \rangle' = \mu\langle\ ,\ \rangle$.

Lemma 6.5. *The definition of Chern form does not depend on the choice of local holomorphic orthonormal frame field. Furthermore, two homothetic Kähler metrics have the same Chern forms.*

Proof. One holomorphic orthonormal frame is transformed into another by means of a unitary matrix A. Let $\Xi = (\Xi_{ij})$ be the matrix of complex curvature forms corresponding to the first holomorphic orthonormal frame. Then the matrix corresponding to the second holomorphic orthonormal frame is $A^{-1}\Xi A$. Using well-known properties of the determinant, we compute

$$\begin{aligned} \det\left(I - \frac{1}{2\pi\sqrt{-1}} A^{-1}\Xi A\right) &= \det\left(A^{-1}\left(I - \frac{1}{2\pi\sqrt{-1}}\Xi\right)A\right) \qquad (6.5)\\ &= \det\left(I - \frac{1}{2\pi\sqrt{-1}}\Xi\right). \end{aligned}$$

In (6.5) we used only the existence of A^{-1} and not the fact that A was a unitary matrix. So, the same proof shows that the Chern forms are invariant under a homothetic change of metric. □

In [Chern3] Chern showed that the total Chern form γ of a Kähler manifold is closed. (In general γ need not be a harmonic form, however.)

Definition. *Let M be a compact Kähler manifold with total Chern form γ. The de Rham*[2] *cohomology class $[\gamma]$ is called the* **total Chern class** *of M, and $[\gamma_i] \in H^{2i}(M, \mathbb{R})$ is called the i^{th}* **Chern class** *of M.*

Not only is the total Chern class $[\gamma]$ invariant under homothetic changes of metrics (see, for example, [KN, volume 2, page 307]), but also it is invariant under all changes of metrics in the class of metrics compatible with the given almost complex structure.[3]

The first and second Chern forms of a Kähler manifold M are given by

$$2\pi\gamma_1(v,w) = \sqrt{-1}\sum_{i=1}^{n}\Xi_{ii}(v,w) = \sum_{i=1}^{n}\Omega_{ii^*}(v,w), \qquad (6.6)$$

$$4\pi^2\gamma_2(v,w,x,y) = -\sum_{ij=1}^{n}\left\{\Xi_{ii}\wedge\Xi_{jj} - \Xi_{ij}\wedge\Xi_{ji}\right\}(v,w,x,y) \qquad (6.7)$$

[2] Georges de Rham (1903–1990). Swiss Mathematician, who proved the fundamental theorem that expresses the cohomology of a compact differentiable manifold in terms of differential forms. De Rham was professor at the Universities of Geneva and Lausanne.

[3] Properly speaking, γ is the total Chern form of the tangent bundle of M. In the study of tubes we shall not need to deal with the Chern classes of vector bundles other than the tangent bundle.

for tangent vectors v, w, x, y to M. In fact, the first Chern form γ_1 is a variant of the Ricci curvature; from (6.3) and (6.6) it follows that

$$\rho(x,y) = 2\pi\gamma_1(x, Jy) \tag{6.8}$$

for tangent vectors x, y to M. Thus the other Chern forms can be thought of roughly as higher degree analogs of the Ricci curvature.

To find convenient expressions for the Chern forms, we let $\Omega_{i_1\cdots i_{2k}}$ denote the Pfaffian of the matrix $\Omega = (\Omega_{i_p i_q})_{1\le p,q\le 2k}$. So, according to Section 5.3, we have

$$\Omega_{i_1\cdots i_{2k}} = \frac{1}{2^k k!}\sum_{\sigma\in\mathfrak{S}_{2k}} \varepsilon_\sigma\, \Omega_{i_{\sigma(1)}i_{\sigma(2)}}\wedge\ldots\wedge\Omega_{i_{\sigma(2k-1)}i_{\sigma(2k)}}. \tag{6.9}$$

For Kähler manifolds there is a close relation between the Pfaffians $\Omega_{i_1\cdots i_{2k}}$ and the Chern forms γ_k. To find it, we need a lemma that relates the Pfaffian to the determinant for certain matrices.

Lemma 6.6. *Let A and B be commuting $k\times k$ matrices with entries from a commutative ring. Assume that ${}^tA = -A$ and ${}^tB = B$. Then*

$$\mathrm{Pf}\begin{pmatrix} A & B \\ -B & A\end{pmatrix} = \left(\sqrt{-1}\right)^{k^2}\det\left(A - \sqrt{-1}\,B\right). \tag{6.10}$$

Proof. Since $\begin{pmatrix} A & B \\ -B & A\end{pmatrix}$ is a skew symmetric matrix, its Pfaffian is well defined. Moreover, the square of the Pfaffian is the determinant, and so using elementary operations that preserve the determinant, we compute

$$\begin{aligned}
\left(\mathrm{Pf}\begin{pmatrix} A & B \\ -B & A\end{pmatrix}\right)^2 &= \det\begin{pmatrix} A & B \\ -B & A\end{pmatrix}\\
&= \det\begin{pmatrix} A-\sqrt{-1}\,B & B+\sqrt{-1}\,A \\ -B & A\end{pmatrix}\\
&= \det\begin{pmatrix} A-\sqrt{-1}\,B & 0 \\ -B & A+\sqrt{-1}\,B\end{pmatrix}\\
&= \det(A-\sqrt{-1}\,B)\det(A+\sqrt{-1}\,B)\\
&= |\det(A-\sqrt{-1}\,B)|^2.
\end{aligned}$$

Thus both $\mathrm{Pf}\begin{pmatrix} A & B \\ -B & A\end{pmatrix}$ and $|\det(A-\sqrt{-1}\,B)|$ are square roots of $\det\begin{pmatrix} A & B \\ -B & A\end{pmatrix}$, and so we can write

$$\mathrm{Pf}\begin{pmatrix} A & B \\ -B & A\end{pmatrix} = \theta(k)\det(A-\sqrt{-1}\,B), \tag{6.11}$$

where θ is a function of k alone, and $|\theta(k)| = 1$.

To compute $\theta(k)$, we need only evaluate both sides of (6.11) on some well chosen $k \times k$ matrix. We choose $A = 0$ and $B = I$. The right-hand side of (6.11) is easily found to be $\theta(k)(-\sqrt{-1}\,)^k$, but the left-hand side is more difficult to compute. There exists a $2k \times 2k$ orthogonal matrix P (in fact, a permutation matrix), such that

$$\det(P) = (-1)^{\frac{1}{2}k(k-1)}$$

and

$$P\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}{}^tP \;=\; \begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}. \tag{6.12}$$

(For example, for $k = 2$ we can take

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.)$$

From the definition (5.10) it follows that the Pfaffian of the right-hand side of (6.12) is 1. Hence from (5.13) and (6.12) we obtain

$$\begin{aligned} 1 \;&=\; \mathrm{Pf}\left(P\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}{}^tP\right) \qquad (6.13) \\ &=\; \mathrm{Pf}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\det(P) \\ &=\; (-1)^{\frac{1}{2}k(k-1)}\,\mathrm{Pf}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \end{aligned}$$

From (6.11) and (6.13) we get $\theta(k) = (\sqrt{-1}\,)^{k^2}$; consequently, (6.10) holds. □

Lemma 6.7. *The Chern forms of a Kähler manifold M are related to the real curvature forms of M by the formulas*

$$(2\pi)^k \gamma_k = \sum_{i_1 < \cdots < i_k} \Omega_{i_1 i_1^* \cdots i_k i_k^*}. \tag{6.14}$$

for $k = 1, 2, \ldots, n$.

Proof. The definition (6.4) of the Chern form γ_k can be written more explicitly as

$$(2\pi)^k\gamma_k = (\sqrt{-1})^k \sum_{i_1<\cdots<i_k} \det\begin{pmatrix} \Xi_{i_1 i_1} & \cdots & \Xi_{i_1 i_k} \\ \vdots & \ddots & \vdots \\ \Xi_{i_k i_1} & \cdots & \Xi_{i_k i_k} \end{pmatrix}. \tag{6.15}$$

Lemma 6.6 implies that

$$\det\begin{pmatrix} \Xi_{i_1 i_1} & \cdots & \Xi_{i_1 i_k} \\ \vdots & \ddots & \vdots \\ \Xi_{i_k i_1} & \cdots & \Xi_{i_k i_k} \end{pmatrix} = (\sqrt{-1})^{-k^2}\,\mathrm{Pf}\begin{pmatrix} \Omega_{i_1 i_1} & \cdots & \Omega_{i_1 i_k^*} \\ \vdots & \ddots & \vdots \\ -\Omega_{i_1 i_k^*} & \cdots & \Omega_{i_k i_k} \end{pmatrix}. \tag{6.16}$$

Thus from (6.15) and (6.16) we get

$$\begin{aligned}(2\pi)^k\gamma_k &= (-1)^{\frac{1}{2}k(k-1)} \sum_{i_1<\cdots<i_k} \Omega_{i_1\cdots i_k i_1^*\cdots i_k^*} \\ &= \sum_{i_1<\cdots<i_k} \Omega_{i_1 i_1^*\cdots i_k i_k^*}.\end{aligned}$$

□

The top Chern form is particularly important. From the definition of Euler form on page 76 we get:

Corollary 6.8. *The top Chern form γ_n of a Kähler manifold M is the Euler form:*

$$(2\pi)^n\gamma_n = \mathrm{Pf}(\Omega). \tag{6.17}$$

Thus we have another description of the Chern forms: they are generalizations of the Euler form. The additional structure inherent in a Kähler manifold allows us to define not just the $2n$-form $\mathrm{Pf}(\Omega)$, but the other Chern forms as well.

6.2 Spaces of Constant Holomorphic Sectional Curvature

A 2-dimensional subspace Π_m of a tangent space M_m to an almost complex manifold M is called a **holomorphic section**, provided there are tangent vectors x and Jx in M_m that span Π_m. Not every 2-dimensional subspace Π_m of M_m is holomorphic, but it is always possible to choose a basis of a general Π_m of the form $\{x, aJx+y\}$ for some real number a, where $\langle x,y\rangle = 0$. The fact that it is possible to single out the holomorphic sections among the 2-dimensional subspaces has important consequences for the study of the curvature of Kähler manifolds.

The **holomorphic sectional curvature** K_{hol} of an almost Hermitian manifold M is the restriction of the ordinary sectional curvature to holomorphic sections

of tangent spaces. Thus K_{hol} can be regarded as a function that assigns a real number $K_{\mathrm{hol}}(x)$ to each unit tangent vector x to M. We extend the definition of K_{hol} to all tangent vectors x to M by the formula

$$K_{\mathrm{hol}}(x) = R_{xJxxJx}.$$

Definition. *An almost Hermitian manifold is said to have* **constant holomorphic sectional curvature**, *provided there is a constant λ such that*

$$K_{\mathrm{hol}}(x) = 4\lambda\|x\|^4 \tag{6.18}$$

for all vectors x tangent to the manifold. Of particular interest are the Kähler manifolds of constant holomorphic sectional curvature. A Kähler manifold of constant holomorphic sectional curvature will be denoted by $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ and will be called a **space of constant holomorphic sectional curvature**.

Lemma 6.9. *The curvature of a space of constant holomorphic sectional curvature $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ is given by*

$$\begin{aligned} R_{wxyz} = \lambda\Big\{ &\langle w,y\rangle\langle x,z\rangle - \langle w,z\rangle\langle x,y\rangle \\ &+ \langle Jw,y\rangle\langle Jx,z\rangle - \langle Jw,z\rangle\langle Jx,y\rangle + 2\langle Jw,x\rangle\langle Jy,z\rangle\Big\} \end{aligned} \tag{6.19}$$

for tangent vectors w, x, y, z to $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$.

Proof. **(Following [BG].)** Let us put

$$B(x,y) = R_{xyxy}$$

for tangent vectors x and y. It is possible to express the complete curvature tensor R in terms of the components $B(x,y)$, and these components are in turn expressible in terms of K_{hol}. More exact formulas are:

$$\begin{aligned} 32B(x,y) &= 3K_{\mathrm{hol}}(x+Jy) + 3K_{\mathrm{hol}}(x-Jy) - K_{\mathrm{hol}}(x+y) \\ &\quad - K_{\mathrm{hol}}(x-y) - 4K_{\mathrm{hol}}(x) - 4K_{\mathrm{hol}}(y), \end{aligned} \tag{6.20}$$

$$\begin{aligned} 12R_{wxyz} &= B(w+y, x+z) + B(w-y, x-z) \\ &\quad - B(w+z, x+y) - B(w-z, x-y) \\ &\quad - 2B(w,z) - 2B(x,y) + 2B(w,y) + 2B(x,z). \end{aligned} \tag{6.21}$$

A long but straightforward calculation using (6.18), (6.20) and (6.21) yields (6.19). ☐

Corollary 6.10. *The sectional curvature of a Kähler manifold $\mathbb{K}^n_{\text{hol}}(\lambda)$ of constant holomorphic sectional curvature 4λ is given by*

$$K_{xy} = \lambda\left(1 + \frac{3\langle Jx, y\rangle^2}{\|x\|^2\|y\|^2 - \langle x, y\rangle^2}\right)$$

for linearly independent x and y. Consequently, the sectional curvature of $\mathbb{K}^n_{\text{hol}}(\lambda)$ varies between λ and 4λ.

Proof. The sectional curvature is easily computed using (6.19). □

6.3 Locally Symmetric Spaces and Their Compatible Submanifolds

An important class of manifolds that includes both spheres and complex projective spaces is the class of locally symmetric spaces. By definition a Riemannian manifold M is called **locally symmetric**, provided the **covariant derivative** ∇R of the curvature tensor of M vanishes. Here ∇R is defined as follows:

$$\begin{aligned}\nabla_V(R)_{WXYZ} &= V\left(R_{WXYZ}\right) - R_{\nabla_V WXYZ} \\ &\quad - R_{W\nabla_V XYZ} - R_{WX\nabla_V YZ} - R_{WXY\nabla_V Z}\end{aligned} \tag{6.22}$$

for $V, W, X, Y, Z \in \mathfrak{X}(M)$.

The curvature tensor field of any Riemannian manifold satisfies the **second Bianchi identity**

$$\underset{VWX}{\mathfrak{S}} \nabla_V(R)_{WXYZ} = 0. \tag{6.23}$$

For a nice proof of (6.23) avoiding coordinate systems see [Nomizu, page 62]. Despite the similarity between the second Bianchi identity and the condition $\nabla R = 0$, the latter is infinitely stronger. On the one hand, it is out of the question to classify all Riemannian manifolds; but the classification of complete simply connected locally symmetric spaces amounts to the classification of globally symmetric spaces, and this was first done by É. Cartan. For the classification of symmetric spaces and a complete list see, for example, [He].

We already know some examples of locally symmetric spaces:

Lemma 6.11. *A space $\mathbb{K}^n(\lambda)$ of constant sectional curvature λ is locally symmetric.*

(The proof is similar to that of Lemma 6.15 given further on. It is also easy to see that the product of locally symmetric spaces is locally symmetric.)

Let P be a submanifold of a Riemannian manifold M with normal bundle $\boldsymbol{\nu}$. For $(p, u) \in \boldsymbol{\nu}$ we define a linear transformation $R_u \colon M_p \longrightarrow M_p$ by

$$R_u x = R_{ux} u$$

for $x \in M_p$, where R is the curvature operator of M. Then $\langle R_u x, y \rangle = \langle R_u y, x \rangle$, so that R_u is a symmetric linear transformation. If ξ is a unit-speed geodesic in M, we define $R(t) \colon M_{\xi(t)} \longrightarrow M_{\xi(t)}$ by

$$R(t)x = R_{\xi'(t)x}\xi'(t).$$

(This is a slight generalization of the definition of $R(t)$ that we gave in Section 3.1 of Chapter 3.)

The proof of the following lemma is an elementary consequence of the definitions.

Lemma 6.12. *Let M be a locally symmetric space. Then the eigenvalues of $R(t)$ along a unit-speed geodesic $\xi(t)$ are constant, and the corresponding eigenvectors can be chosen parallel.*

In order to study tubes in a locally symmetric space M, we need to single out those submanifolds of M whose second fundamental forms are closely related to the curvature operator of M. To this end, let P be a submanifold of a Riemannian manifold M and let $p \in P$. For $u \in P_p^\perp$ the Weingarten map $T_u \colon P_p \longrightarrow P_p$ has a natural extension $T_u \oplus I \colon M_p \longrightarrow M_p$, where $I \colon P_p^\perp \longrightarrow P_p^\perp$ is the identity map.

Definition. *We say that a submanifold P of a Riemannian manifold M is* **compatible** *with M, provided for all $p \in P$ and for each unit vector $u \in P_p^\perp$ the linear transformations $T_u \oplus I$ and R_u are simultaneously diagonalizable.*

Let us note some examples of compatible submanifolds.

Lemma 6.13. *Any submanifold of a space $\mathbb{K}^n(\lambda)$ of constant sectional curvature is compatible with $\mathbb{K}^n(\lambda)$. Any point in any Riemannian manifold M is compatible with M.*

Proof. For a space $\mathbb{K}^n(\lambda)$ the operator R_u is given by $R_u = \lambda I$ for any unit tangent vector u. Therefore, R_u and $T_u \oplus I$ are always simultaneously diagonalizable. Since R_u and I are simultaneously diagonalizable, the last statement follows. $\square$

A submanifold of a locally symmetric space M more complicated than $\mathbb{K}^n(\lambda)$ need not be compatible with M. For example, a curve that sits diagonally in the product $\mathbb{K}^n(\lambda) \times \mathbb{K}^m(\lambda)$ of two spaces of constant curvature λ will not be compatible with $\mathbb{K}^n(\lambda) \times \mathbb{K}^m(\lambda)$ when $\lambda \neq 0$.

Now we come to an important fact concerning tubular hypersurfaces about a compatible submanifold of a locally symmetric space.

Theorem 6.14. *Let M be a locally symmetric space, and let P be a submanifold that is compatible with M. Then:*

(i) *the eigenvectors of the shape operators of the tubular hypersurfaces about P can be chosen parallel along geodesics normal to P;*

(ii) *any tubular hypersurface about P is also compatible with M.*

Proof. Let ξ be a unit-speed geodesic in M that meets P normally at $p = \xi(0)$. By Lemma 6.12 the eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ of $R(t)$ are constant functions of t along ξ. If $\{E_1, \ldots, E_n\}$ are the corresponding eigenvectors, then by Lemma 6.12 they can be chosen parallel along ξ. Let $\widetilde{\kappa}_\alpha$ be the solution to the differential equation $\widetilde{\kappa}'_\alpha = \widetilde{\kappa}^2_\alpha + \lambda_\alpha$ with the initial conditions

$$\widetilde{\kappa}_\alpha(0) = \begin{cases} \kappa_\alpha(0) & \text{for } 1 \leq \alpha \leq q, \\ 0 & \text{for } \alpha = q+1, \\ -\infty & \text{for } q+2 \leq \alpha \leq n, \end{cases}$$

where $\kappa_\alpha(0)$ are the eigenvalues of $T_{\xi'(0)}$, and write

$$\widetilde{S} = \begin{pmatrix} \widetilde{\kappa}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \widetilde{\kappa}_n \end{pmatrix}.$$

Then we have $\widetilde{S}'(t) = \widetilde{S}(t)^2 + R(t)$. But by (3.8) we also have $S'(t) = S(t)^2 + R(t)$. Since $S(0) = \widetilde{S}(0)$, it follows that $S(t)$ and $\widetilde{S}(t)$ coincide for all t. Thus $\{E_1, \ldots, E_n\}$ are the eigenvectors of $S(t)$, so part (i) follows. We also see that $S(t) \oplus I$ and $R(t)$ are simultaneously diagonalizable for all t, so part (ii) follows as well. $\square$

Now we turn to the complex case. The first observation is the following:

Corollary 6.15. *A Kähler manifold M with constant holomorphic sectional curvature is locally symmetric.*

Proof. Equation (6.19) can be written in terms of vector fields as

$$\begin{aligned} R_{WXYZ} &= \lambda\Big\{\langle W,Y\rangle\langle X,Z\rangle - \langle W,Z\rangle\langle X,Y\rangle \\ &+ \langle JW,Y\rangle\langle JX,Z\rangle - \langle JW,Z\rangle\langle JX,Y\rangle + 2\langle JW,X\rangle\langle JY,Z\rangle\Big\} \end{aligned} \tag{6.24}$$

for $W, X, Y, Z \in \mathfrak{X}(M)$. When we take the covariant derivatives of both sides of (6.24), we get a formula for ∇R. A long but straightforward calculation using (6.22), together with the assumptions that λ is constant and $\nabla J = 0$, shows that $\nabla R = 0$. $\square$

It should be remarked that there is a curvature condition of interest that is potentially weaker than that of constant holomorphic sectional curvature: one can require that the holomorphic sectional curvature be pointwise constant, that is $K_{\text{hol}}(x) = 4\lambda(m)\|x\|^4$ for $x \in M_m$ where $\lambda \in \mathfrak{F}(M)$. But if M is Kählerian and $\dim(M) \geq 4$, such a function λ is necessarily constant. The proof uses the second Bianchi identity and is a variant of F. Schur's Theorem from Riemannian geometry (see, for example, [KN, page 168, volume 2]). (However, there are complex manifolds for which the holomorphic sectional curvature is pointwise constant, but varies from point to point [GV2].)

It follows from the classification of symmetric spaces (see, for example, [He]) that a space $\mathbb{K}^n_{\text{hol}}(\lambda)$ of constant holomorphic sectional curvature is locally isometric to one of the following spaces:

complex Euclidean space	$\mathbb{C}^n$	$(\lambda = 0)$,
complex projective space	$\mathbb{C}P^n(\lambda)$	$(\lambda > 0)$,
complex hyperbolic space	$\mathbb{C}H^n(\lambda)$	$(\lambda < 0)$.

Here $\mathbb{C}^n$ is complex Euclidean space with a flat Kähler metric, $\mathbb{C}P^n(\lambda)$ is complex projective space and $\mathbb{C}H^n(\lambda)$ is complex hyperbolic space. We shall have more to say about complex projective space in the next section. Complex hyperbolic space can be realized as the unit ball $\{\, (z_1, \ldots, z_n) \;\in \mathbb{C}^n \mid \sum_{j=1}^n |z_j|^2 < 1\}$ with a constant multiple of the Bergman metric. (For details see, for example, [He].)

There are many compatible submanifolds of $\mathbb{K}^n_{\text{hol}}(\lambda)$.

Lemma 6.16. *Any complex submanifold P of a space $\mathbb{K}^n_{\text{hol}}(\lambda)$ of a constant holomorphic sectional curvature is compatible with $\mathbb{K}^n_{\text{hol}}(\lambda)$.*

Proof. Let $p \in \mathbb{K}^n_{\text{hol}}(\lambda)$. For any unit vector u tangent to $\mathbb{K}^n_{\text{hol}}(\lambda)$ at p the eigenvalues of the R_u are 4λ (with multiplicity 1), λ (with multiplicity $2n-2$), and 0 (with multiplicity 1). So, the eigenspace decomposition of $\mathbb{K}^n_{\text{hol}}(\lambda)_p$ with respect to R_u is

$$\mathbb{K}^n_{\text{hol}}(\lambda)_p = \{u\} \oplus \{Ju\} \oplus \{u, Ju\}^{\perp}.$$

Furthermore, $Ju \in P_p^{\perp}$ because P is a complex submanifold of $\mathbb{K}^n_{\text{hol}}(\lambda)$, and therefore we have the decomposition

$$\mathbb{K}^n_{\text{hol}}(\lambda)_p = P_p \oplus \{u\} \oplus \{Ju\} \oplus (P_p^{\perp} \cap \{u, Ju\}^{\perp}). \tag{6.25}$$

Each of the subspaces on the right-hand side of (6.25) is invariant under $T_u \oplus I$. It follows that $T_u \oplus I$ and R_u are simultaneously diagonalizable. ☐

Corollary 6.17. *Let P be a complex submanifold of $\mathbb{K}^n_{\text{hol}}(\lambda)$. Then:*

(i) *each tubular hypersurface P_t about P is compatible with $\mathbb{K}^n_{\text{hol}}(\lambda)$;*

(ii) *if N is the vector field that is the unit normal to each tubular hypersurface P_t about P, then JN is a principal curvature vector of the shape operator $S(t)$ of P_t.*

Proof. Since $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ is locally symmetric, it follows from part (ii) of Theorem 6.14 that P_t is compatible with $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$. Furthermore, JN starts off as an eigenvector of $S(0)$ and is parallel; so it remains an eigenvector of $S(t)$. □

Remark. Conversely, Cecil and Ryan [CR2] have shown that if a real hypersurface Q of $\mathbb{C}P^n(\lambda)$ has the property that JN is an eigenvector of the shape operator at all points of Q, then with an additional assumption Q is a tubular hypersurface of a complex submanifold of $\mathbb{C}P^n(\lambda)$. Recently, A.A. Borisenko [Boris] has shown that, with no additional assumption, Q is a tubular hypersurface around an irreducible algebraic variety.

6.4 Geodesic Balls in a Space $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ of Constant Holomorphic Sectional Curvature

The formula for the volume of a geodesic ball turns out to be surprisingly simple for a Kähler manifold $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ of constant holomorphic sectional curvature 4λ. To see how much more complicated volumes of geodesic balls in a space $\mathbb{K}^n(\lambda)$ are, see the discussion before Lemma 3.16. The derivative of formula (6.26) below occurs in Study's paper [Study].

Lemma 6.18. *The volume of a geodesic ball in a space $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ of constant holomorphic sectional curvature 4λ is given by*

$$V_m^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}(r) = \frac{1}{n!}\left(\frac{\pi}{\lambda}\right)^n \left(\sin(r\sqrt{\lambda})\right)^{2n}. \tag{6.26}$$

Proof. Given $u \in \mathbb{K}^n_{\mathrm{hol}}(\lambda)_m$ with $\|u\| = 1$, let ξ be a unit-speed geodesic with $\xi(0) = m$ and $\xi'(0) = u$. Let $\{e_1, \dots, Je_n\}$ be a holomorphic orthonormal frame which diagonalizes R_u on the tangent space $\mathbb{K}^n_{\mathrm{hol}}(\lambda)_m$, and put $u = e_1$. Along the geodesic ξ we choose a parallel holomorphic orthonormal frame field $t \longmapsto \{E_1(t), \dots, JE_n(t)\}$ which coincides with $\{e_1, \dots, Je_n\}$ at m. Since points of M are trivially compatible with M, each $E_i(t)$ is an eigenvector of both $R(t)$ and $S(t)$, where $S(t)$ is the shape operator at $\xi(t)$ of the geodesic sphere

$$\{\, m' \in \mathbb{K}^n_{\mathrm{hol}}(\lambda) \mid \mathrm{distance}(m, m') = t \,\}.$$

So along ξ the Riccati equations for the principal curvature functions of the geodesic spheres are:

$$\begin{cases} \kappa_2' &= \kappa_2^2 + 4\lambda, \\ \kappa_i' &= \kappa_i^2 + \lambda \qquad (3 \le i \le 2n), \end{cases} \tag{6.27}$$

with the initial conditions $\kappa_2(0) = \cdots = \kappa_{2n}(0) = -\infty$. Equations (6.27) can be solved just as in the proofs of Lemmas 3.14 and 3.16:

$$\kappa_2(t) = \frac{-2\sqrt{\lambda}}{\tan(2t\sqrt{\lambda})},$$

$$\kappa_i(t) = \frac{-\sqrt{\lambda}}{\tan(t\sqrt{\lambda})} \qquad (3 \le i \le 2n).$$

Thus by Theorem 3.11

$$\begin{aligned}\frac{\vartheta_u'(t)}{\vartheta_u(t)} &= -\frac{2n-1}{t} + \frac{2\sqrt{\lambda}}{\tan(2t\sqrt{\lambda})} + \frac{(2n-2)\sqrt{\lambda}}{\tan(t\sqrt{\lambda})} \\ &= -\frac{2n-1}{t} + \frac{2\sqrt{\lambda}\cos(2t\sqrt{\lambda})}{\sin(2t\sqrt{\lambda})} + \frac{(2n-2)\sqrt{\lambda}\cos(t\sqrt{\lambda})}{\sin(t\sqrt{\lambda})}.\end{aligned}$$

Using the fact that $\vartheta_u(0) = 1$ and integrating, we obtain

$$\begin{aligned}\log\vartheta_u(t) &= -(2n-1)\log t + \log(\sin(2t\sqrt{\lambda})) + (2n-2)\log(\sin(t\sqrt{\lambda})) \\ &\quad - \lim_{t\to 0^+}\Big\{ -(2n-1)\log t + \log(\sin(2t\sqrt{\lambda})) + (2n-2)\log(\sin(t\sqrt{\lambda}))\Big\} \\ &= \log\left(\tfrac{1}{2}t^{-2n+1}\sin(2t\sqrt{\lambda})(\sin(t\sqrt{\lambda}))^{2n-2}\lambda^{-n+\frac{1}{2}}\right),\end{aligned}$$

or

$$t^{2n-1}\vartheta_u(t) = \big(\sin(t\sqrt{\lambda})\big)^{2n-1}\cos(t\sqrt{\lambda})\lambda^{-n+\frac{1}{2}}. \tag{6.28}$$

Therefore, from (6.28) and Lemmas 3.12 and 3.13 we have

$$\begin{aligned}V_m^{\mathbb{K}^n_{\text{hol}}(\lambda)}(r) &= \int_0^r \int_{S^{2n-1}(1)} \big(\sin(t\sqrt{\lambda})\big)^{2n-1}\cos(t\sqrt{\lambda})\lambda^{-n+\frac{1}{2}}\,du\,dt \\ &= \frac{2\pi^n}{\Gamma(n)}\int_0^r \big(\sin(t\sqrt{\lambda})\big)^{2n-1}\cos(t\sqrt{\lambda})\lambda^{-n+\frac{1}{2}}\,dt \\ &= \frac{1}{n!}\left(\frac{\pi}{\lambda}\right)^n \big(\sin(r\sqrt{\lambda})\big)^{2n}.\end{aligned}$$

□

An important fact that arises from the proof of Lemma 6.18 is that $\vartheta_u(t)$ is independent of u. This is a very special property of the space $\mathbb{K}^n_{\text{hol}}(\lambda)$. (A manifold M with the property that $\vartheta_u(t)$ is the same for unit tangent vectors u to M is called a **harmonic space**. In fact, until 1992 the only known examples of harmonic spaces were Euclidean space, spheres, complex projective spaces, quaternionic projective spaces, the Cayley plane, and the negative curvature analogs of these spaces. E. Damek and F. Ricci [DaRi] found examples of non-symmetric harmonic spaces.

For more details on harmonic spaces see [Bessel, Chapter 6], [RWW], [Sza], [BCG], [Vanh] and [BTV].) We shall exploit the independence of $\vartheta_u(t)$ on u to compute the volume of complex projective space $\mathbb{C}P^n(\lambda)$ in the next section.

Thus all the known examples of harmonic spaces are homogeneous of a very special kind: the isotropy representation acts transitively on the unit sphere of the tangent space of any point. We shall need the following lemma in the next section:

Lemma 6.19. *Let M be a homogeneous Riemannian manifold such that the isotropy representation acts transitively on the unit sphere of each tangent space. Then the distance from a point p to its first conjugate point in any direction is independent of p and the direction.*

Proof. Since for each $p \in M$ the infinitesimal change of volume function $\vartheta_u(t)$ is independent of u, so is the distance from p to the first zero of ϑ_u along the geodesic $t \longmapsto \exp_m(tu)$. □

6.5 Complex Projective Space $\mathbb{C}P^n(\lambda)$

In this section we shall mention a few important facts about complex projective space $\mathbb{C}P^n(\lambda)$ that are useful for the study of tubes. As a complex manifold, complex projective space is the space of complex lines in $\mathbb{C}^{n+1}$. Alternatively, let $\sim$ be the equivalence relation on $S^{2n+1}(1)$ given by $(z_0, \dots, z_n) \sim (w_0, \dots, w_n)$ if and only if there is a complex number a with $|a| = 1$ such that $z_i = aw_i$ for $0 \leq i \leq n$. It can be shown that $\mathbb{C}P^n(1)$ is the set of equivalence classes of $\sim$. Then $\mathbb{C}P^n(1)$ is a manifold and the projection $\pi: S^{2n+1}(1) \longrightarrow \mathbb{C}P^n(1)$ is differentiable and surjective. So, π is a submersion, because each tangent map of π is onto.

For the study of tubes it is important that $\mathbb{C}P^n(1)$ is not just a manifold, but a Riemannian manifold, in fact a Kähler manifold. The (canonical) almost complex structure on $\mathbb{C}P^n(1)$ can be obtained as follows. Let $\tilde{J}$ denote the usual almost complex structure of $\mathbb{C}^{n+1}$ and $\tilde{N}$ the unit normal to $S^{2n+1}(1)$. Then $\tilde{J}\tilde{N}$ is a unit tangent vector field to $S^{2n+1}(1)$ and is in the kernel of the tangent map π_{*p} for each $p \in S^{2n+1}(1)$. The almost complex structure J of $\mathbb{C}P^n(1)$ is defined by $J\pi_{*p}x = \pi_{*p}(\tilde{J}x)$ for $p \in S^{2n+1}(1)$ and $x \in (\operatorname{kernel} \pi_{*p})^{\perp}$. It is not hard to check that this J is well defined.

There are several methods to introduce the canonical Fubini[4] -Study metric [Fubini], [Study]. One of the easiest ways is to require the projection $\pi: S^{2n+1}(1) \longrightarrow \mathbb{C}P^n(1)$ to be a Riemannian submersion. This means each tangent map π_{*p} maps $(\operatorname{kernel} \pi_{*p})^{\perp}$ isometrically onto $\mathbb{C}P^n(1)_{\pi(p)}$. The resulting metric (often called the **Fubini-Study metric**) is Kählerian with respect to J. Then $\mathbb{C}P^n(\lambda)$ is $\mathbb{C}P^n(1)$ with

[4] Guido Fubini (1879–1943). Italian mathematician. Born in Venice, Fubini studied at the Scuola Normale in Pisa, where he worked with Dini and Bianchi. Fubini was professor in Turin until he was forced to leave Italy in 1939 because of anti-semitism.

the Fubini-Study[5] metric multiplied by λ^{-2} where $\lambda > 0$. The verification that this metric is Kählerian and has constant holomorphic sectional curvature 4λ can be accomplished by means of the submersion equations (see, for example, [Gr1], [ON2] and [Besse2, Chapter 9]). Consequently, $\mathbb{C}P^n(\lambda)$ is a symmetric space (and in fact a rank 1 symmetric space). For this reason $\mathbb{C}P^n(\lambda)$ plays roughly the same role among Kähler manifolds that the sphere plays among real manifolds.

Corollary 6.20. *The volume of $\mathbb{C}P^n(\lambda)$ is given by*

$$\text{volume}(\mathbb{C}P^n(\lambda)) = \frac{1}{n!}\Big(\frac{\pi}{\lambda}\Big)^n. \tag{6.29}$$

Proof. From Lemmas 6.18 and 3.13 we have

$$\mathrm{A}_m^{\mathbb{C}P^n(\lambda)}(t) = \frac{2\pi^n}{(n-1)!\lambda^{n-\frac{1}{2}}}\big(\sin(t\sqrt{\lambda})\big)^{2n-1}\cos(t\sqrt{\lambda}). \tag{6.30}$$

Then $A_m^{\mathbb{C}P^n(\lambda)}(t)$ vanishes for $t = 0$ trivially. But by Lemma 6.19, the next time that it vanishes, say for $t = t_0$, is when the geodesic ball

$$\{\, m' \in \mathbb{C}P^n(\lambda) \mid \text{distance}(m', m) < t \,\}$$

exactly covers $\mathbb{C}P^n(\lambda)$ except for a set of measure zero. This is because $\vartheta_u(t)$ is the same in all directions u.

Now $\cos(t_0\sqrt{\lambda}) = 0$ implies $\sin(t_0\sqrt{\lambda}) = 1$. Therefore,

$$\text{volume}(\mathbb{C}P^n(\lambda)) = \mathrm{V}_m^{\mathbb{C}P^n(\lambda)}(t_0) = \frac{1}{n!}\Big(\frac{\pi}{\lambda}\Big)^n. \qquad \square$$

6.6 The Chern Forms of a Kähler Manifold $\mathbb{K}^n_{\rm hol}(\lambda)$ of Constant Holomorphic Sectional Curvature

Because of its prominence among Kähler manifolds, it is important to compute the Chern forms of $\mathbb{C}P^n(\lambda)$. We know from Lemma 6.5 that the total Chern form of $\mathbb{C}P^n(\lambda)$ (in contrast to the Kähler form) does not depend on the choice of λ.

There are topological methods for computing the *total Chern class* of $\mathbb{C}P^n(\lambda)$ (see, for example, [MS]). However, in Theorem 6.24 we shall obtain a more precise result by computing the *total Chern form* of $\mathbb{C}P^n(\lambda)$. This has the added advantage that, at the same time, we obtain a formula for the total Chern form of a Kähler manifold of constant negative holomorphic sectional curvature.

Before beginning the calculation of the Chern forms of $\mathbb{K}^n_{\rm hol}(\lambda)$, it is useful to have some preliminary facts about matrices of complex differential forms.

[5] Eduard Study (1862–1930). German mathematician, professor at the University of Bonn. His major work, **Geometrie der Dynamen**, in which he considered Euclidean kinematics and the mechanics of rigid bodies, was published in 1903. Study was a leader in the geometry of complex numbers and wrote important papers on isotropic curves. Since any minimal surface arises from an isotropic curve, Study's work has applications in minimal surface theory.

Lemma 6.21. *Suppose $\alpha_1, \ldots, \alpha_c$ are complex 1-forms (on any differentiable manifold). Then*

$$\det\begin{pmatrix} \alpha_1 \wedge \bar{\alpha}_1 & \ldots & \alpha_1 \wedge \bar{\alpha}_c \\ \vdots & \ddots & \vdots \\ \alpha_c \wedge \bar{\alpha}_1 & \ldots & \alpha_c \wedge \bar{\alpha}_c \end{pmatrix} = c!\, \alpha_1 \wedge \bar{\alpha}_1 \wedge \ldots \wedge \alpha_c \wedge \bar{\alpha}_c.$$

Proof. Since $\varepsilon_\sigma\, \bar{\alpha}_{\sigma(1)} \wedge \ldots \wedge \bar{\alpha}_{\sigma(c)} = \bar{\alpha}_1 \wedge \ldots \wedge \bar{\alpha}_c$, it follows that

$$\varepsilon_\sigma\, \alpha_1 \wedge \bar{\alpha}_{\sigma(1)} \wedge \ldots \wedge \alpha_c \wedge \bar{\alpha}_{\sigma(c)} = \alpha_1 \wedge \bar{\alpha}_1 \wedge \ldots \wedge \alpha_c \wedge \bar{\alpha}_c.$$

Write $a_{ij} = \alpha_i \wedge \bar{\alpha}_j$. Then by definition of the determinant we have

$$\begin{aligned} \det(a_{ij}) &= \sum_{\sigma \in \mathfrak{S}_c} \varepsilon_\sigma\, a_{1\sigma(1)} \cdots a_{c\sigma(c)} \\ &= \sum_{\sigma \in \mathfrak{S}_c} \varepsilon_\sigma\, \alpha_1 \wedge \bar{\alpha}_{\sigma(1)} \wedge \ldots \wedge \alpha_c \wedge \bar{\alpha}_{\sigma(c)} \\ &= c!\, \alpha_1 \wedge \bar{\alpha}_1 \wedge \ldots \wedge \alpha_c \wedge \bar{\alpha}_c. \end{aligned}$$

□

Next, for any almost Hermitian manifold M let $\{E_1, \ldots, JE_n\}$ be a holomorphic orthonormal frame field, and let $\theta_1, \theta_{1^*}, \ldots, \theta_n, \theta_{n^*}$ be the corresponding dual 1-forms. In order to compute the complex curvature forms, it is reasonable to introduce the **complex dual 1-forms** ϕ_a. We put

$$\phi_a = \theta_a + \sqrt{-1}\, \theta_{a^*}, \tag{6.31}$$

for $1 \leq a \leq n$.

Lemma 6.22. *For any almost Hermitian manifold M the Kähler form F of M is related to the dual 1-forms by the equations*

$$F = \frac{1}{2}\sum_{a=1}^{2n} \theta_a \wedge \theta_{a^*} = \frac{\sqrt{-1}}{2}\sum_{a=1}^{n} \phi_a \wedge \bar{\phi}_a. \tag{6.32}$$

Furthermore, we have the identity

$$\sum_{a_1 \ldots a_c = 1}^{n} \det\begin{pmatrix} \phi_{a_1} \wedge \bar{\phi}_{a_1} & \ldots & \phi_{a_1} \wedge \bar{\phi}_{a_c} \\ \vdots & \ddots & \vdots \\ \phi_{a_c} \wedge \bar{\phi}_{a_1} & \ldots & \phi_{a_c} \wedge \bar{\phi}_{a_c} \end{pmatrix} = c!\,(-2\sqrt{-1}\, F)^c. \tag{6.33}$$

Proof. Equations (6.32) are easy consequences of the definitions, and (6.33) follows from (6.32) and Lemma 6.21. □

Next we compute the curvature forms and complex curvature forms of $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$. Notice that the formulas are not very different from the formula (5.20) for the curvature forms of a hypersurface in Euclidean space.

Lemma 6.23. *The real and complex curvature forms of* $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ *are related to the dual* 1*-forms by the formulas*

$$\Omega_{\alpha\beta} = \lambda\Big\{\theta_\alpha \wedge \theta_\beta + \theta_{\alpha^*} \wedge \theta_{\beta^*} - 2\delta_{\alpha\beta^*}F\Big\}, \tag{6.34}$$

$$\Xi_{ab} = \lambda\Big\{\phi_a \wedge \bar{\phi}_b - 2\sqrt{-1}\,\delta_{ab}F\Big\}, \tag{6.35}$$

for $1 \leq \alpha, \beta \leq 2n$ *and* $1 \leq a, b \leq n$.

Proof. From (6.19) we obtain

$$\begin{aligned}
\Omega_{\alpha\beta}(X,Y) &= \langle R_{XY}E_\alpha, E_\beta\rangle \\
&= \lambda\Big\{\langle X, E_\alpha\rangle\langle Y, E_\beta\rangle - \langle X, E_\beta\rangle\langle Y, E_\alpha\rangle \\
&\quad + \langle JX, E_\alpha\rangle\langle JY, E_\beta\rangle - \langle JX, E_\beta\rangle\langle JY, E_\alpha\rangle + 2\langle JX, Y\rangle\langle JE_\alpha, E_\beta\rangle\Big\} \\
&= \lambda\Big\{\theta_\alpha(X)\theta_\beta(Y) - \theta_\alpha(Y)\theta_\beta(X) \\
&\quad + \theta_{\alpha^*}(X)\theta_{\beta^*}(Y) - \theta_{\alpha^*}(Y)\theta_{\beta^*}(X) - 2\delta_{\alpha\beta^*}F(X,Y)\Big\} \\
&= \lambda\Big\{\theta_\alpha \wedge \theta_\beta + \theta_{\alpha^*} \wedge \theta_{\beta^*} - 2\delta_{\alpha\beta^*}F\Big\}(X,Y).
\end{aligned}$$

Since X and Y are arbitrary elements of $\mathfrak{X}(M)$, we get (6.34).

Also, from (6.34) it follows that for $1 \leq a, b \leq n$

$$\begin{aligned}
\Xi_{ab} &= \Omega_{ab} - \sqrt{-1}\,\Omega_{ab^*} \\
&= \lambda\Big\{\theta_a \wedge (\theta_b - \sqrt{-1}\,\theta_{b^*}) + \sqrt{-1}\,\theta_{a^*} \wedge (\theta_b - \sqrt{-1}\,\theta_{b^*}) - 2\delta_{ab}\sqrt{-1}\,F\Big\} \\
&= \lambda\Big\{\phi_a \wedge \bar{\phi}_b - 2\delta_{ab}\sqrt{-1}\,F\Big\}.
\end{aligned}$$

□

Now we are ready to compute the Chern forms of $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$. In fact, we compute them simultaneously by computing the total Chern form.

Theorem 6.24. *The total Chern form of a space $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ of constant holomorphic sectional curvature is given by*

$$\gamma = \left(1 + \frac{\lambda}{\pi}F\right)^{n+1}. \tag{6.36}$$

Proof. Using the rule for expanding determinants (compare equation (4.17), page 63), we have

$$\begin{aligned}
\gamma &= \det\left(\delta_{ab} + \frac{\lambda\sqrt{-1}}{2\pi}\left(\phi_a \wedge \bar{\phi}_b - 2\sqrt{-1}\,\delta_{ab}F\right)\right) \\
&= \det\left(\left(1 + \frac{\lambda}{\pi}F\right)\delta_{ab} + \frac{\lambda\sqrt{-1}}{2\pi}\phi_a \wedge \bar{\phi}_b\right) \\
&= \sum_{c=0}^{n}\left(1 + \frac{\lambda}{\pi}F\right)^{n-c}\left(\frac{\lambda\sqrt{-1}}{2\pi}\right)^{c}\psi_c,
\end{aligned}$$

where

$$c!\,\psi_c = \sum_{a_1\ldots a_c=1}^{n} \det\begin{pmatrix} \phi_{a_1} \wedge \bar{\phi}_{a_1} & \cdots & \phi_{a_1} \wedge \bar{\phi}_{a_c} \\ \vdots & \ddots & \vdots \\ \phi_{a_c} \wedge \bar{\phi}_{a_1} & \cdots & \phi_{a_c} \wedge \bar{\phi}_{a_c} \end{pmatrix}.$$

By (6.33) we have $\psi_c = \left(-2\sqrt{-1}\,F\right)^c$, and so

$$\begin{aligned}
\gamma &= \sum_{c=0}^{n}\left(1 + \frac{\lambda}{\pi}F\right)^{n-c}\left(\frac{\lambda\sqrt{-1}}{2\pi}\right)^{c}\left(-2\sqrt{-1}\,F\right)^{c} \qquad (6.37)\\
&= \sum_{c=0}^{n}\left(1 + \frac{\lambda}{\pi}F\right)^{n-c}\left(\frac{\lambda}{\pi}F\right)^{c}.
\end{aligned}$$

The right-hand side of (6.37) is a geometric series. Using formal manipulation, it can be rewritten as

$$\frac{\left(1 + \frac{\lambda}{\pi}F\right)^{n+1} - \left(\frac{\lambda}{\pi}F\right)^{n+1}}{\left(1 + \frac{\lambda}{\pi}F\right) - \left(\frac{\lambda}{\pi}F\right)} = \left(1 + \frac{\lambda}{\pi}F\right)^{n+1}. \tag{6.38}$$

Then (6.36) follows from (6.37) and (6.38). □

Corollary 6.25. *The i^{th} Chern form γ_i of $\mathbb{K}^n_{\text{hol}}(\lambda)$ is given by*

$$\gamma_i = \binom{n+1}{i}\left(\frac{\lambda}{\pi}F\right)^i. \tag{6.39}$$

Proof. Equation (6.39) is proved by equating forms of degree $2i$ on the left-hand side of (6.36) with the corresponding forms on the right-hand side. □

Remarks. (1) Since the total Chern form does not depend on λ, the apparent dependence of the right-hand side of (6.36) and (6.39) on λ must be canceled out by the dependence of the Kähler form F on λ.

(2) When the right-hand side of (6.36) is expanded by the binomial theorem, the term of highest degree is zero because its degree is larger than the dimension of $\mathbb{K}^n_{\text{hol}}(\lambda)$. Thus for example, the total Chern form of $\mathbb{C}P^2(\lambda)$ is

$$1 + 3\left(\frac{\lambda}{\pi}F\right) + 3\left(\frac{\lambda}{\pi}F\right)^2.$$

6.7 Kähler Submanifolds and Wirtinger's Inequality

Strong use will be made in this section of the assumption that metrics are positive definite. Strictly speaking, Wirtinger's[6] inequality is independent of Weyl's Tube Formula. However, we discuss it in this section in order to provide some perspective.

Let M be any Riemannian manifold and P an isometrically immersed submanifold of dimension p. The **mean curvature vector field**[7] H of a Riemannian submanifold is given by

$$H = \sum_{a=1}^{p} T_{E_a} E_a$$

for any local orthonormal frame $\{E_1, \ldots, E_p\}$ on P, where T is the second fundamental form of the immersion of P in M. Then P is said to be a **minimal variety**, provided H vanishes identically.

Now, let M be a Kähler manifold with almost complex structure J and metric $\langle\ ,\ \rangle$. We say that a submanifold P of M is a **Kähler submanifold**, provided that whenever x is a tangent vector to P, so is Jx. Clearly the restriction J to P is an almost complex structure on P which we call the **induced almost complex structure**; we denote it by the same letter J.

[6]Wilhelm Wirtinger (1865–1945). Austrian mathematician. Born in Ybbs, Austria, Wirtinger studied at the University of Vienna, at which university he received his doctorate in 1887 and his habilitation in 1890. He was greatly influenced by Felix Klein, with whom he studied at Berlin and Göttingen. Wirtinger taught at the Universities of Vienna and Innsbruck. His range of mathematics was exceptional. He wrote on theta functions, geometry, algebra, number theory, invariants, relativity and statistics. Among the mathematicians who Wirtinger taught while he held the chair at Vienna were Schreier, Gödel, Radon and Taussky-Todd.

[7]This is a slight change of notation from Chapter 1, where H was a function.

Lemma 6.26. *Suppose P is a Kähler submanifold of M. Then the induced Riemannian metric and almost complex structure make P into a Kähler manifold. Furthermore, the mean curvature vector field of P vanishes.*

Proof. Let ∇^M and ∇^P denote the covariant derivatives of M and P. Then we have

$$\nabla^M = \nabla^P + T, \tag{6.40}$$

where T is the second fundamental form of P. From (6.40) follows

$$0 = \nabla^M_A(J) = \nabla^M_A J - J\nabla^M_A = \cdots = \nabla^P_A(J) + T_A J - JT_A \tag{6.41}$$

for $A \in \mathfrak{X}(P)$. Since $\nabla^P_A(J)B$ is tangent to P and $T_A JB - JT_A B$ is normal to P for $A, B \in \mathfrak{X}(P)$, it follows that both must vanish. Therefore, P is Kählerian, and we have

$$T_{JA} JB = JT_{JA} B = JT_B JA = -T_B A = -T_A B.$$

Hence the mean curvature vector H of P is given by

$$H = \sum_{a=1}^{p} \left(T_{E_a} E_a + T_{JE_a} JE_a \right) = 0. \qquad \square$$

Thus any Kähler submanifold is a minimal variety in the sense that its mean curvature vector vanishes. However, we shall now show that a Kähler submanifold is a minimal variety in the stronger sense of being volume minimizing among all manifolds homologous to it. First, we make what seems to be a trivial observation, but we shall see that it has important consequences.

Lemma 6.27. *The Kähler form F of a Kähler submanifold P is the restriction to P of the Kähler form of M.*

Proof. This is an obvious consequence of the fact that the metric and almost complex structure of P are the restrictions to P of the metric and almost complex structure of M. $\square$

Another important fact is

Lemma 6.28. $(1/n!)F^n$ *is one of the two Riemannian volume elements of a Kähler manifold M. Hence it determines an orientation of M, the* **canonical orientation.**

Proof. Let $\{E_1, \ldots, JE_n\}$ be any local holomorphic orthonormal frame and put $A_{ij} = F(E_i, E_j)$ for $(1 \leq i, j \leq 2n)$. Since F is antisymmetric, so is the matrix $S = (A_{ij})$. Moreover,

$$\begin{aligned} \frac{1}{n!} F^n(E_1, \ldots, JE_n) &= \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \varepsilon_\sigma \, A_{i_{\sigma(1)} i_{\sigma(2)}} \wedge \ldots \wedge A_{i_{\sigma(2n-1)} i_{\sigma(2n)}} \\ &= \sum{}' \varepsilon_\sigma A_{i_{\sigma(1)} i_{\sigma(2)}} \wedge \ldots \wedge A_{i_{\sigma(2n-1)} i_{\sigma(2n)}}, \end{aligned}$$

where $\sum'$ is the sum over all permutations σ such that $\sigma(2i-1) < \sigma(2i)$ for $1 \leq i \leq n$ and $\sigma(1) < \sigma(3) < \cdots < \sigma(2n-1)$. When $i < j$, we have $A_{ij} = 0$ unless $j = i^*$; also $A_{ii^*} = 1$. Hence

$$\sum{}' \varepsilon_\sigma A_{i_{\sigma(1)} i_{\sigma(2)}} \wedge \ldots \wedge A_{i_{\sigma(2n-1)} i_{\sigma(2n)}} = A_{11^*} \cdots A_{nn^*} = 1.$$

Thus $(1/n!)F^n$ is the Riemannian volume element having the same orientation as any local holomorphic orthonormal frame. □

From Lemmas 6.27 and 6.28 we get

Corollary 6.29. *For a Kähler submanifold P of M, the volume element corresponding to the canonical orientation of P is $(1/p!)F^p$, where $2p$ is the real dimension of P.*

Using Lemmas 6.27, 6.28 and Corollary 6.29, we derive a formula for the volume of a Kähler submanifold P of M in terms of the Kähler form of M. Let $\mathbf{e}^F$ denote the formal power series

$$\sum_{k=0}^{\infty} \frac{F^k}{k!}.$$

Theorem 6.30. *The volume of any Kähler submanifold P with compact closure can be expressed as an integral of a power of the Kähler form F of M. More precisely, we have*

$$\operatorname{volume}(P) = \frac{1}{p!} \int_P F^p = \int_P \mathbf{e}^F, \tag{6.42}$$

where $2p$ is the real dimension of P.

Proof. Since$(1/p!)F^p$ is the volume element of P, the first integral on the right-hand side of (6.42) reduces to the definition of the volume of P. When $\mathbf{e}^F$ is expanded formally in a power series and all terms not of degree $2p$ are discarded, the second integral in (6.42) reduces to the first.[8] □

Let P and P' be compact q-dimensional submanifolds of a manifold M. We say that P and P' are **homologous**, provided that there exists a $(q+1)$-chain B such that $P - P' = \partial B$. The next theorem is due to Wirtinger [Wir], although the simple proof given below is due to Federer [Fd2] and is based on Theorem 6.30. (See also [HL].)

Theorem 6.31. (Wirtinger's Inequality.) *Let P be a Kähler submanifold of M with finite volume, and let P' be any submanifold which is homologous to P. Then*

$$\operatorname{volume}(P) \leq \operatorname{volume}(P'). \tag{6.43}$$

[8]Of course, it is not necessary to introduce the notation $\mathbf{e}^F$. But notice that using $\mathbf{e}^F$, the formula for the volume of a submanifold can be written without reference to the dimension of the submanifold.

Proof. Let B be a manifold (of dimension $2p+1$) such that $\partial B = P - P'$. By Stokes' Theorem we have

$$0 = \int_B dF^p = \int_{P-P'} F^p = \int_P F^p - \int_{P'} F^p. \tag{6.44}$$

Let ω' denote the volume element of P'. Since ω' and $(1/p!)F^p$ are volume elements, they both have norm 1. Therefore, by the Cauchy-Schwarz Inequality

$$\left|\left\langle \omega', \frac{1}{p!}F^p \right\rangle\right| \leq \|\omega'\| \left\| \frac{1}{p!}F^p \right\| = 1. \tag{6.45}$$

From (6.44) and (6.45), we obtain

$$\int_P \mathbf{e}^F = \int_{P'} \mathbf{e}^F = \int_{P'} \left\langle \omega', \frac{1}{p!}F^p \right\rangle \omega' \leq \int_{P'} \omega' = \text{volume}(P'). \tag{6.46}$$

Now (6.43) follows from Theorem 6.30 and (6.46). □

6.8 The Homology and Cohomology of Complex Projective Space $\mathbb{C}P^n(\lambda)$

We recall the well-known fact that

Theorem 6.32. *The real cohomology of* $\mathbb{C}P^n(\lambda)$ *is (as a ring)*

$$H^*(\mathbb{C}P^n(\lambda), \mathbb{R}) = \left\{ [F]^k \;\middle|\; k = 0, 1, 2, \dots \right\} \Big/ \{ [F]^{n+1} = 0 \}.$$

In fact, there is a more precise result.

Theorem 6.33. *The integral cohomology of* $\mathbb{C}P^n(\lambda)$ *is (as a ring)*

$$H^*(\mathbb{C}P^n(\lambda), \mathbb{Z}) = \left\{ \left[\frac{\lambda}{\pi}F\right]^k \;\middle|\; k = 0, 1, 2, \dots \right\} \Big/ \left\{ \left[\frac{\lambda}{\pi}F\right]^{n+1} = 0 \right\}.$$

Proof. The generator for the homology group $H_k(\mathbb{C}P^n(\lambda), \mathbb{Z})$ is the submanifold $\mathbb{C}P^k(\lambda)$. Let x^k denote the generator for the integral cohomology group $H^k(\mathbb{C}P^n(\lambda), \mathbb{Z})$. Then x^k also belongs to the real cohomology group $H^k(\mathbb{C}P^n(\lambda), \mathbb{R})$ and hence $x^k = a_k[F]^k$ for some a_k. To determine a_k, it suffices to integrate x^k over the cycle $\mathbb{C}P^k(\lambda)$ and use formula (6.29) for the volume of $\mathbb{C}P^k(\lambda)$. We have

$$\begin{aligned} 1 &= x^k[\mathbb{C}P^k(\lambda)] = a_k[F]^k[\mathbb{C}P^k(\lambda)] \\ &= a_k \int_{\mathbb{C}P^k(\lambda)} F^k \\ &= a_k\, k!\ \text{volume}\left(\mathbb{C}P^k(\lambda)\right) = a_k \left(\frac{\pi}{\lambda}\right)^k. \end{aligned}$$

Hence $a_k = \left(\frac{\lambda}{\pi}\right)^k$, and the theorem follows. □

6.9 Chern Numbers

What is the analog for a compact Kähler manifold M of dimension $2n$ of the Euler characteristic? Since the top Chern form γ_n coincides with the Euler form, it is clear that

$$\chi(M) = \int_M \gamma_n,$$

but what about the other Chern forms? We must have a $2n$-form to integrate over M, so the question is how to get a $2n$-form using the various Chern forms. Suppose, for example, that the dimension of M is 6. Then the products of Chern forms that have degree 6 are γ_1^3, $\gamma_2 \wedge \gamma_1$ and γ_3. Each of these can be integrated over M. This leads to the following definition.

Definition. *The* **Chern number** $c_{i_1} c_{i_2} \cdots c_{i_k}(M)$ *is defined as follows. Let* $(i_1, \ldots, i_k)$ *be a sequence of numbers (not necessarily distinct) such that* $i_1 + i_2 + \cdots + i_k = n$. *Then*

$$c_{i_1} c_{i_2} \cdots c_{i_k}(M) = \int_M \gamma_{i_1} \wedge \ldots \wedge \gamma_{i_k}.$$

Thus the three possible Chern numbers of a 6-dimensional Kähler manifold M are $c_1^3(M)$, $c_1 c_2(M)$ and $c_3(M)$. (See also problem 6.2.)

Since we have formulas for the total Chern form and volume of $\mathbb{C}P^n(\lambda)$, we can find all of its Chern numbers.

Corollary 6.34. *For complex projective space* $\mathbb{C}P^n(\lambda)$, *we have*

$$c_{i_1} \cdots c_{i_k}\big(\mathbb{C}P^n(\lambda)\big) = \binom{n+1}{i_1} \cdots \binom{n+1}{i_k}.$$

Proof. It follows from Corollary 6.25 that

$$\begin{aligned}
c_{i_1} \cdots c_{i_k}(\mathbb{C}P^n(\lambda)) &= \int_{\mathbb{C}P^n(\lambda)} \gamma_{i_1} \wedge \ldots \wedge \gamma_{i_k} \\
&= \binom{n+1}{i_1} \cdots \binom{n+1}{i_k} \int_{\mathbb{C}P^n(\lambda)} \left(\frac{\lambda}{\pi} F\right)^n \\
&= \binom{n+1}{i_1} \cdots \binom{n+1}{i_k} \left(\frac{\lambda}{\pi}\right)^n n! \text{ volume}\big(\mathbb{C}P^n(\lambda)\big) \\
&= \binom{n+1}{i_1} \cdots \binom{n+1}{i_k}.
\end{aligned}$$

□

Notice that the Chern numbers of complex projective space are all integers. In fact, for any Kähler manifold M the Chern numbers are interesting integers that generalize the Euler characteristic.

6.10 Complex Hypersurfaces of Complex Projective Space $\mathbb{C}P^{n+1}(\lambda)$

A **complex hypersurface** of a complex manifold is a complex submanifold of real codimension 2. Just as in the case of real hypersurfaces of real manifolds, complex hypersurfaces are in some ways simpler than general complex submanifolds. Particularly important are complex hypersurfaces in $\mathbb{C}P^{n+1}(\lambda)$.

Consider the homogeneous equation

$$\sum_{i=0}^{n+1} a_i z_i^d = 0, \tag{6.47}$$

where d is a positive integer and the a_i are complex numbers. Since (6.47) continues to hold when each z_i is replaced by λz_i, it makes sense not only on $\mathbb{C}^{n+2}$, but also on $\mathbb{C}P^{n+1}(\lambda)$. More generally, let $P_d(z_0, \ldots, z_{n+1})$ be any homogeneous polynomial of degree d and consider the set of zeros of P_d:

$$M_d(\lambda) = \{\, [z_0, \ldots, z_{n+1}] \in \mathbb{C}P^{n+1}(\lambda) \mid P_d(z_0, \ldots, z_{n+1}) = 0 \,\}.$$

Then $M_d(\lambda)$ is, in fact, a complex submanifold of $\mathbb{C}P^{n+1}(\lambda)$; it is called a **complex hypersurface of degree** d. It is easy to see that any complex hypersurface of degree 1 in $\mathbb{C}P^{n+1}(\lambda)$ is isometric to $\mathbb{C}P^{n}(\lambda)$. However, for $d > 1$ the Riemannian geometry of $M_d(\lambda)$ (for example, the curvature) will in general depend on the polynomial P_d (as well as on λ). However, there are many Riemannian invariants that are independent of P_d. In particular, in Chapter 7 we shall show that the volume of a tube of radius r about a complex hypersurface $M_d(\lambda)$ depends only on λ and the degree d, and is otherwise independent of the polynomial P_d.

In the present section we write down the Chern classes and compute the Chern numbers of a complex hypersurface of degree d.

Theorem 6.35. *The total Chern class of $M_d(\lambda) \subset \mathbb{C}P^{n+1}(\lambda)$ is given by*

$$[\gamma(M_d(\lambda))] = \left[\frac{\left(1 + \dfrac{\lambda}{\pi} F\right)^{n+2}}{1 + \dfrac{\lambda d}{\pi} F} \right]. \tag{6.48}$$

For a proof see [Hi, page 159].

Note that when $d = 1$, formula (6.48) follows from Theorem 6.24. However, the formula

$$\gamma(\mathbb{C}P^{n}(\lambda)) = \left(1 + \frac{\lambda}{\pi} F\right)^{n+1}$$

is more precise because it is a formula for the total Chern form of $\mathbb{C}P^{n}(\lambda)$, whereas (6.48) is only a formula for the total Chern class.

Corollary 6.36. *The first three Chern classes of a hypersurface $M_d(\lambda) \subset \mathbb{C}P^{n+1}(\lambda)$ are given by*

$$\begin{aligned}
[\gamma_1(M_d(\lambda))] &= (n+2-d)\left[\frac{\lambda}{\pi}F\right], \\
[\gamma_2(M_d(\lambda))] &= \left(\frac{1}{2}(n+2)(n+1) - d(n+2) + d^2\right)\left[\frac{\lambda}{\pi}F\right]^2, \\
[\gamma_3(M_d(\lambda))] &= \left(\frac{1}{6}(n+2)(n+1)n - \frac{d}{2}(n+2)(n+1)\right. \\
&\qquad\qquad \left. + d^2(n+2) - d^3\right)\left[\frac{\lambda}{\pi}F\right]^3.
\end{aligned}$$

More generally,

$$[\gamma_k(M_d(\lambda))] = \left(\sum_{l=0}^{k}\binom{n+2}{l}(-d)^{k-l}\right)\left[\frac{\lambda}{\pi}F\right]^k. \tag{6.49}$$

Proof. To get (6.49), we expand the right-hand side of (6.48) and equate forms of like degree with the left-hand side. □

Notice that when $d = 1$ in (6.48) or (6.49), we recover the formula for the Chern classes of $\mathbb{C}P^n(\lambda)$.

In addition to the inequality of [Wir], Wirtinger also found a simple formula for the volume of a complete intersection in $\mathbb{C}P^{n+1}(\lambda)$ (see problem 6.5). A special case is the formula for the volume of a complex hypersurface of degree d. We now give a proof of this formula.

Theorem 6.37. *The volume of a complex hypersurface $M_d(\lambda) \subset \mathbb{C}P^{n+1}(\lambda)$ of degree d is given by*

$$\text{volume}(M_d(\lambda)) = d \cdot \text{volume}\left(\mathbb{C}P^n(\lambda)\right) = \frac{d}{n!}\left(\frac{\pi}{\lambda}\right)^n. \tag{6.50}$$

Proof. $M_d(\lambda)$ is a cycle of $\mathbb{C}P^{n+1}(\lambda)$ because it is a compact submanifold without boundary. Hence it gives rise to a homology class

$$[M_d(\lambda)] \in H_n(\mathbb{C}P^{n+1}(\lambda), \mathbb{Z}).$$

Therefore, $[M_d(\lambda)]$ must be an integral multiple of the generator $[\mathbb{C}P^n(\lambda)] \in H_n(\mathbb{C}P^{n+1}(\lambda), \mathbb{Z})$. A more exact formulas is:

$$[M_d(\lambda)] = d \cdot [\mathbb{C}P^n(\lambda)].$$

Now we use the fact (Corollary 6.29) that the volume form of $M_d(\lambda)$ is the restriction of $(1/n!)F^n$ to $M_d(\lambda)$, where F is the Kähler form of $\mathbb{C}P^{n+1}(\lambda)$. Therefore,

we have

$$\begin{aligned}\text{volume}\left(M_d(\lambda)\right) &= \frac{1}{n!}\int_{M_d(\lambda)} F^n = \frac{1}{n!}[F^n]\cdot[M_d(\lambda)] \\ &= \frac{1}{n!}[F^n]\cdot d\cdot[\mathbb{C}P^n(\lambda)] = d\cdot\text{volume}\left(\mathbb{C}P^n(\lambda)\right) = \frac{d}{n!}\left(\frac{\pi}{\lambda}\right)^n. \qquad \square\end{aligned}$$

Next we compute some of the Chern numbers of hypersurfaces of degree d. First, we need the following elementary fact.

Lemma 6.38. *Let M be any Kähler manifold of real dimension $2n$ whose Chern classes satisfy the generalized Einstein condition*

$$c_i = [\gamma_i] = f(i)\left[\frac{\lambda}{\pi}F\right]^i.$$

Let $i_1,\ldots,i_k$ be such that $i_1+i_2+\cdots+i_k=n$. Then

$$c_{i_1}\cdots c_{i_k}(M) = \left[\frac{\lambda}{\pi}\right]^n f(i_1)f(i_2)\cdots f(i_k)\cdot n!\cdot\text{volume}(M).$$

In particular,

$$c_{i_1}\cdots c_{i_k}(M_d(\lambda)) = d\cdot f(i_1)f(i_2)\cdots f(i_k).$$

The proof is obvious.

The values of the $f(i)$ for a complex hypersurface $M_d(\lambda)$ can be determined by means of Corollary 6.36 together with the formula (6.48) for the volume of $M_d(\lambda)$. For the Chern numbers of hypersurfaces of low degree, see problem 6.3. Here are some more general formulas.

Lemma 6.39. *We have*

$$\begin{aligned} c_1^n(M_d(\lambda)) &= (n+2-d)^n d, \\ c_n(M_d(\lambda)) &= \chi(M_d(\lambda)) = d\sum_{k=0}^{n}\binom{n+2}{k}(-d)^{n-k}, \\ &= \frac{1}{d}\left\{(1-d)^{n+2}+(n+2)d-1\right\}. \end{aligned}$$

6.11 Problems

6.1 Compute explicitly the Chern numbers of complex projective space $\mathbb{C}P^n(\lambda)$ for $n\le 4$ (see Table I).

6.2 Find all possibilities for the Chern numbers of an almost complex manifold M of complex dimension less than or equal to six (see Table II).

	Table I
Complex dimension	Chern numbers of complex projective spaces
1	$c_1(\mathbb{C}P^1(\lambda)) = 2$
2	$c_1^2(\mathbb{C}P^2(\lambda)) = 9, \quad c_2(\mathbb{C}P^2(\lambda)) = 3$
3	$c_1^3(\mathbb{C}P^3(\lambda)) = 64, \quad c_2c_1(\mathbb{C}P^3(\lambda)) = 24,$ $c_3(\mathbb{C}P^3(\lambda)) = 4$
4	$c_1^4(\mathbb{C}P^4(\lambda)) = 625, \quad c_2c_1^2(\mathbb{C}P^4(\lambda)) = 250,$ $c_2^2(\mathbb{C}P^4(\lambda)) = 100, \quad c_1c_3(\mathbb{C}P^4(\lambda)) = 50,$ $c_4(\mathbb{C}P^4(\lambda)) = 5$

6.3 Compute the Chern numbers of a complex hypersurface $M_d(\lambda)$ of degree d in low dimensions (see Table III).

6.4 Show that the Ricci and scalar curvatures of a space $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ of constant holomorphic sectional curvature 4λ are given by

$$\rho(X, Y) = 2(n+1)\lambda\langle X, Y\rangle \quad \text{and} \quad \tau = 4n(n+1)\lambda$$

for $X, Y \in \mathfrak{X}(\mathbb{K}^n_{\mathrm{hol}}(\lambda))$.

6.5 A **complete intersection** $P_{a_1\ldots a_r}(\lambda) \subset \mathbb{C}P^{n+r}(\lambda)$ is defined to be the set of simultaneous zeros of polynomials of degrees $a_1, \ldots, a_r$. Generalize the formula for the volume of a complex hypersurface in $\mathbb{C}P^{n+1}(\lambda)$ by showing that

$$\mathrm{volume}(P_{a_1\ldots a_r}(\lambda)) = \frac{a_1 \cdots a_r \pi^n}{n!\lambda^n}. \tag{6.51}$$

Hence the volume of a complete intersection depends only on the degrees of the polynomials.

	Table II
Complex dimension	Possible Chern numbers
1	$c_1(M)$
2	$c_1^2(M), \quad c_2(M)$
3	$c_1^3(M), \quad c_2c_1(M), \quad c_3(M)$
4	$c_1^4(M), \quad c_2c_1^2(M),$ $c_2^2(M), \quad c_1c_3(M), \quad c_4(M)$
5	$c_1^5(M), \quad c_1^3c_2(M), \quad c_1^2c_3(M),$ $c_2^2c_1(M), \quad c_3c_2(M), \quad c_1c_4(M), \quad c_5(M)$
6	$c_1^6(M), \quad c_1^4c_2(M), \quad c_1^3c_3(M), \quad c_1^2c_4(M),$ $c_2^2c_1^2(M), \quad c_1c_5(M), \quad c_1c_2c_3(M), \quad c_2^3(M),$ $c_3^2(M), \quad c_2c_4(M), \quad c_6(M)$

6.6 Show that the total Chern class of the complete intersection $P_{a_1\ldots a_r}(\lambda) \subset \mathbb{C}P^{n+r}(\lambda)$ is given by

$$[\gamma(P_{a_1\ldots a_r}(\lambda))] = \left[\frac{\left(1+\dfrac{\lambda}{\pi}F\right)^{n+r+1}}{\prod_{c=1}^{r}\left(1+\dfrac{a_c\lambda}{\pi}F\right)}\right].$$

	Table III
Complex dimension of $M_d(\lambda)$	Chern numbers of the complex hypersurfaces $M_d(\lambda)$
1	$c_1(M_d(\lambda)) = (3-d)d$
2	$c_1^2(M_d(\lambda)) = (4-d)^2 d$, $c_2(M_d(\lambda)) = (6-4d+d^2)d$
3	$c_1^3(M_d(\lambda)) = (5-d)^3 d$, $c_2c_1(M_d(\lambda)) = (5-d)(10-5d+d^2)d$, $c_3(M_d(\lambda)) = (10-10d+5d^2-d^3)d$
4	$c_1^4(M_d(\lambda)) = (6-d)^4 d$, $c_2c_1^2(M_d(\lambda)) = (6-d)^2(15-6d+d^2)d$, $c_3c_1(M_d(\lambda)) = (6-d)(20-15d+6d^2-d^3)d$, $c_2^2(M_d(\lambda)) = (15-6d+d^2)^2 d$, $c_4(M_d(\lambda)) = (15-20d+15d^2-6d^3+d^4)d$

6.7 What goes wrong if one tries to mimic the theory of complex hypersurfaces in $\mathbb{C}P^n(\lambda)$ by studying quaternionic hypersurfaces in quaternionic projective space?

Chapter 7

The Tube Formula in the Complex Case

For a complex submanifold P of a complex Euclidean space $\mathbb{C}^n$, we shall give a considerably simplified version of Weyl's Tube Formula. It turns out that all the coefficients $k_{2c}(P)$ have a topological nature similar to that of the top coefficient. We shall derive explicit formulas for the $k_{2c}(P)$ as integrals involving the Chern forms and Kähler form of P. To carry out this program, we require generalizations of the Bianchi and Kähler identities to double forms; these we give in Section 7.1. We use these identities in Section 7.2 to derive formula (7.18) for the volume of a tube about a complex submanifold with compact closure in a complex Euclidean space. We call it the **Complex Weyl Tube Formula**.

Even though the Complex Weyl Tube Formula is far simpler than the general Weyl Tube Formula, the situation is not completely satisfactory because formula (7.18) has little relevance for compact complex submanifolds without boundary in $\mathbb{C}^n$. The problem is that, necessarily, any such submanifold is zero dimensional. This leads us to study complex submanifolds of complex projective space $\mathbb{C}P^n(\lambda)$, in which there are many interesting compact complex submanifolds.

We call the formula for the volume of a tube about a complex submanifold of $\mathbb{C}P^n(\lambda)$ the **Projective Weyl Tube Formula**. We carry out its derivation in two stages. In Section 7.3 we give a tube formula (formula (7.28)) that holds for a complex submanifold P with compact closure in a Kähler manifold $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ of constant holomorphic sectional curvature. The proof of formula (7.28) is a straightforward generalization of Weyl's Tube Formula and involves solving the Riccati equations for the principal curvature functions of the tubular hypersurfaces about P. However, it is not clear from formula (7.28) how the coefficients depend on the Chern forms.

Therefore, in Sections 7.4 and 7.5 we turn to the study of tubes about a complex submanifold P of $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ with two objectives in mind. The first goal is to

rewrite formula (7.28), making the dependence on the Kähler form and total Chern form clear; this we achieve with Theorem 7.20, where we prove the Projective Weyl Tube Formula.

Secondly, we find simple formulas for the volumes of tubes about certain compact complex submanifolds of $\mathbb{C}P^n(\lambda)$. For example, complex hypersurfaces of complex projective space $\mathbb{C}P^n(\lambda)$ are studied in Section 7.6, and a formula for the volume of a tube about a complex hypersurface of degree d resembling the formula for the volume of a geodesic ball is given.

In Section 7.7 we show that the volume of a tube about a complex submanifold of $\mathbb{C}P^n(\lambda)$ is invariant under Kähler deformations. Finally, in Section 7.8 we derive the formula for the volume of a tube about a totally real totally geodesic submanifold of complex projective space.

7.1 Higher Order Curvature Identities

In order to derive a new version of Weyl's Tube Formula in the complex case, it will be necessary to use various curvature identities involving the c^{th} power R^c of the curvature tensor. These are generalizations of well-known curvature identities for the case $c = 1$. We begin with a general Riemannian manifold.

Lemma 7.1. (Generalized first Bianchi identity.) *Let R be the curvature tensor of a Riemannian manifold M, and let $x_1, \dots, x_{2c+1}, y_2, \dots, y_{2c}$ be tangent vectors to M. Then*

$$\sum_{k=1}^{2c+1} (-1)^k R^c(x_1, \dots, \hat{x}_k, \dots, x_{2c+1})(x_k, y_2, \dots, y_{2c}) = 0. \tag{7.1}$$

Proof. We define an operation $\phi \longmapsto \phi'$ mapping the space of double forms of type (p, q) into those of type $(p+1, q-1)$:

$$\phi'(x_1, \dots, x_{p+1})(y_2, \dots, y_q) = \sum_{k=1}^{p+1} (-1)^{k+1} \phi(x_1, \dots, \hat{x}_k, \dots, x_{p+1})(x_k, y_2, \dots, y_q).$$

Then it can be checked that

$$(\phi \wedge \theta)' = \phi' \wedge \theta + (-1)^{p+q} \phi \wedge \theta', \tag{7.2}$$

where ϕ is a double form of type (p, q) and θ is a double form of any type. For the curvature tensor R considered as a double form, the identity

$$R' = 0 \tag{7.3}$$

is just the (first) Bianchi identity (2.19), page 19. So, it follows by induction from (7.2) and (7.3) that

$$(R^c)' = 0. \tag{7.4}$$

Then (7.4) is equivalent to (7.1). □

For more details on the generalized Bianchi identities see [Gr3] and [Th1], [Th2], [Th3].

There is an additional identity for R^c for Kähler manifolds:

Lemma 7.2. (Generalized Kähler identity.) *Let R be the curvature tensor of a Kähler manifold M. Then*

$$R^c(x_{1^*},\dots,x_{2c^*})(y_1,\dots,y_{2c}) = R^c(x_1,\dots,x_{2c})(y_1,\dots,y_{2c}), \tag{7.5}$$

for tangent vectors $x_1,\dots,x_{2c},y_1,\dots,y_{2c}$ to M.

Proof. Formula (7.5) can be proved by induction from the definition of multiplication of double forms (formula (4.2), page 54) and the Kähler identity (6.1). □

Let M be an almost Hermitian manifold of real dimension $2q$. To study a contraction of a tensor field on M, we can use an arbitrary orthonormal basis to do the contraction. Since M is almost Hermitian, we usually choose this basis to be of the form $\{e_1, e_{1^*},\dots,e_q,e_{q^*}\}$. Often we abbreviate e_{a_i} to a_i.

For any almost Hermitian manifold M it is possible to form a contraction of R^c that is, in general, different from the contraction $C^{2c}(R^c)$, which is given by

$$C^{2c}(R^c) = \sum_{a_1\dots a_{2c}=1}^{2q} R^c(a_1,\dots,a_{2c})(a_1,\dots,a_{2c}).$$

The new contraction is:

$$\sum_{\substack{a_1\dots a_c=1\\ b_1\dots b_c=1}}^{2q} R^c(a_1,a_{1^*},\dots,a_c,a_{c^*})(b_1,b_{1^*},\dots,b_c,b_{c^*}).$$

However, for Kähler manifolds, at least, it turns out that this contraction is a multiple of $C^{2c}(R^c)$.

Lemma 7.3. *Let R be a curvature tensor which also satisfies the Kähler identity* (6.1). *Then we have*

$$\sum_{\substack{a_1\dots a_c=1\\ b_1\dots b_c=1}}^{2q} R^c(a_1,a_{1^*},\dots,a_c,a_{c^*})(b_1,b_{1^*},\dots,b_c,b_{c^*}) = \frac{(2^c c!)^2}{(2c)!}C^{2c}(R^c). \tag{7.6}$$

Proof. Let $d_1,\dots,d_{2c}$ be arbitrary elements of a holomorphic orthonormal basis $\{e_1,e_{1^*},\dots,e_q,e_{q^*}\}$, and put $a_i = e_{a_i}$, $b_i = e_{b_i}$ for $1 \le i \le n^*$. Write

$$\begin{aligned} b(i) = \sum_{\substack{a_1\dots a_i=1\\ b_1\dots b_i=1}}^{2q} & R^c(a_1,a_{1^*},\dots,a_i,a_{i^*},d_{2i+1},\dots,d_{2c}) \\ & \cdot (b_1,b_{1^*},\dots,b_i,b_{i^*},d_{2i+1},\dots,d_{2c}). \end{aligned} \tag{7.7}$$

From (7.5) it follows that we can replace each d_k in (7.7) by d_{k*}; thus:

$$b(i) = \sum_{\substack{a_1 \dots a_i = 1 \\ b_1 \dots b_i = 1}}^{2q} R^c(a_1, a_{1^*}, \dots, a_i, a_{i^*}, d_{2i+1^*}, \dots, d_{2c^*}) \cdot (b_1, b_{1^*}, \dots, b_i, b_{i^*}, d_{2i+1}, \dots, d_{2c}). \tag{7.8}$$

At this point we need a somewhat long but straightforward calculation using (7.8) and the generalized first Bianchi identity (7.1). We have

$$\begin{aligned} 0 = \sum_{\substack{a_1 \dots a_i = 1 \\ b_1 \dots b_i = 1}}^{2q} \Big\{ & iR^c(a_{1^*}, a_2, a_{2^*}, \dots, a_i, a_{i^*}, d_{2i+1^*}, \dots, d_{2c^*}, b_1) \\ & \qquad \cdot (a_1, b_{1^*}, \dots, b_i, b_{i^*}, d_{2i+1}, \dots, d_{2c}) \\ & -iR^c(a_1, a_2, a_{2^*}, \dots, a_i, a_{i^*}, d_{2i+1^*}, \dots, d_{2c^*}, b_1) \\ & \qquad \cdot (a_{1^*}, b_{1^*}, \dots, b_i, b_{i^*}, d_{2i+1}, \dots, d_{2c}) \\ & + \sum_{k=1}^{2c-2i} (-1)^{k+1} R^c(a_1, a_{1^*}, \dots, a_i, a_{i^*}, d_{2i+1^*}, \dots, \hat{d}_{2i+k^*}, \dots, d_{2c^*}, b_1) \\ & \qquad \cdot (d_{2i+k^*}, b_{1^*}, \dots, b_i, b_{i^*}, d_{2i+1}, \dots, d_{2c}) \\ & + R^c(a_1, a_{1^*}, \dots, a_i, a_{i^*}, d_{2i+1^*}, \dots, d_{2c^*}) \\ & \qquad \cdot (b_1, b_{1^*}, \dots, b_i, b_{i^*}, d_{2i+1}, \dots, d_{2c}) \Big\} \\ = {} & \textstyle\sum_I + \sum_{II} + \sum_{III} + \sum_{IV} . \end{aligned}$$

Here, $\sum_I$ can be combined with $\sum_{II}$, and $\sum_{III}$ can be combined with $\sum_{IV}$. Thus we get

$$\begin{aligned} 0 = \sum_{\substack{a_1 \dots a_i = 1 \\ b_1 \dots b_i = 1}}^{2q} \Big\{ & -2iR^c(a_2, a_{2^*}, \dots, a_i, a_{i^*}, d_{2i+1^*}, \dots, d_{2c^*}, a_1, b_1) \\ & \qquad \cdot (b_2, b_{2^*}, \dots, b_i, b_{i^*}, d_{2i+1}, \dots, d_{2c}, a_{1^*}, b_{1^*}) \\ & +(2c - 2i + 1) R^c(a_1, a_{1^*}, \dots, a_i, a_{i^*}, d_{2i+1^*}, \dots, d_{2c^*}) \\ & \qquad \cdot (b_1, b_{1^*}, \dots, b_i, b_{i^*}, d_{2i+1}, \dots, d_{2c}) \Big\}, \end{aligned}$$

or more concisely

$$b(i) = \frac{2i}{2c - 2i + 1} b(i-1). \tag{7.9}$$

(Equation (7.9) is a generalization of formula (6.3), page 88, for the Ricci curvature of a Kähler manifold.)

From (7.9) it follows that

$$b(c) = (2c)b(c-1) = (2c)\left(\frac{2c-2}{3}\right)b(c-2)$$
$$= \cdots = \frac{2^c c!}{1\cdot 3\cdots(2c-1)} = \frac{(2^c c!)^2}{(2c)!}b(0),$$

which is the same as (7.6). □

Throughout the rest of this chapter we assume that all almost complex manifolds are Kähler manifolds, unless stated otherwise.

Lemma 7.4. *Let P be a $2q$-dimensional Kähler manifold, and $\phi \in \Lambda^{2c}(P)$, where $\Lambda^{2c}(P)$ denotes the space of $2c$-forms on P. Then*

$$(\phi \wedge F^{q-c})(e_1,\ldots,e_{2q}) = \frac{(q-c)!}{2^c c!}\sum_{a_1\ldots a_c=1}^{q}\phi(a_1,a_{1^*},\ldots,a_c,a_{c^*}). \tag{7.10}$$

Proof. Without loss of generality, we can compute the left-hand side of (7.10) by evaluating $\phi\wedge F^{q-c}$ on a basis of the form $\{e_1,e_{1^*},\ldots,e_q,e_{q^*}\}$. From (4.2) it follows that

$$(\phi\wedge F^{q-c})(1,1^*,\ldots,q,q^*) = \sum_{1\le a_1<\cdots<a_c\le q}\phi(a_1,a_{1^*},\ldots,a_c,a_{c^*})F^{q-c}(a_{c+1},\ldots,a_{q^*}).$$

But $F^{q-c}(a_{c+1},\ldots,a_{q^*}) = (q-c)!$ by the proof of Lemma 6.28, page 106. So we obtain

$$\begin{aligned}(\phi\wedge F^{q-c})(1,1^*,\ldots,q,q^*) &= (q-c)!\sum_{1\le a_1<\cdots<a_c\le q}\phi(a_1,a_{1^*},\ldots,a_c,a_{c^*})\\ &= \frac{(q-c)!}{2^c c!}\sum_{a_1\ldots a_c=1}^{q}\phi(a_1,a_{1^*},\ldots,a_c,a_{c^*}).\end{aligned}$$

Thus we get (7.10). □

Lemma 7.5. *For a Kähler manifold the following relation holds between R^c and the Chern form γ_c:*

$$(2\pi)^c\gamma_c = \frac{1}{2^c(c!)^2}\sum_{a_1\ldots a_c=1}^{2q}R^c(a_1,a_{1^*},\ldots,a_c,a_{c^*}). \tag{7.11}$$

Proof. Lemma 5.5 implies that

$$c!\,\Omega_{a_1a_{1^*}\ldots a_ca_{c^*}} = R^c(a_1,a_{1^*},\ldots,a_c,a_{c^*}). \tag{7.12}$$

Also, from (6.14) we get

$$(2\pi)^c \gamma_c = \sum_{1 \le a_1 < \cdots < a_c \le q} \Omega_{a_1 a_{1^*} \ldots a_c a_{c^*}} \tag{7.13}$$

$$= \frac{1}{2^c c!} \sum_{a_1 \ldots a_c = 1}^{2q} \Omega_{a_1 a_{1^*} \ldots a_c a_{c^*}}.$$

Then (7.11) follows from (7.12) and (7.13). □

Lemma 7.6. *For a Kähler manifold P the $C^{2c}(R^c)$'s and the $k_{2c}(P)$'s are expressed in terms of the Chern forms of P by the formulas*

$$\frac{(q-c)!}{c!(2c)!} C^{2c}(R^c) = (2\pi)^c (\gamma_c \wedge F^{q-c})(e_1, \ldots, e_{2q}); \tag{7.14}$$

$$k_{2c}(P) = \frac{(2\pi)^c}{(q-c)!} \int_P \gamma_c \wedge F^{q-c}. \tag{7.15}$$

Proof. From Lemmas 7.3 and 7.5 we have

$$\frac{(2^c c!)^2}{(2c)!} C^{2c}(R^c) = 2^c (c!)^2 (2\pi)^c \sum_{b_1 \ldots b_c = 1}^{2q} \gamma_c(b_1, b_{1^*}, \ldots, b_c, b_{c^*}). \tag{7.16}$$

On the other hand, it follows from Lemma 7.4 that

$$\sum_{a_1 \ldots a_c = 1}^{2q} \gamma_c(a_1, a_{1^*}, \ldots, a_c, a_{c^*}) = \frac{2^c c!}{(q-c)!} (\gamma_c \wedge F^{q-c})(e_1, \ldots, e_{2q}). \tag{7.17}$$

Combining (7.16) and (7.17), we get (7.14). Then (7.15) results when (7.14) is integrated over P. □

Notice that when $c = q$, equation (7.15) is a special case of equation (5.22), which expresses the top coefficient in Weyl's Tube Formula in terms of the Pfaffian of the curvature tensor.

7.2 Tubes about Complex Submanifolds of $\mathbb{C}^n$

We can now derive the formula for $V_P^{\mathbb{C}^n}(r)$ in terms of the total Chern form and Kähler form of a complex submanifold P of $\mathbb{C}^n$. We shall write $V_P^{\mathbb{C}^n}(r)$ as the integral over P of a certain nonhomogeneous differential form (see formula (7.18) below). In this formula the integral of each differential form whose degree is different from the dimension of P is to be thrown away.

Theorem 7.7. (The Complex Weyl Tube Formula.) *Let P be a topologically embedded complex submanifold of $\mathbb{C}^n$ with compact closure, and assume that* $\exp_\nu \colon \{ (p, v) \in \boldsymbol{\nu} \mid \|v\| \le r \} \longrightarrow T(P, r)$ *is a diffeomorphism. Then*

$$V_P^{\mathbb{C}^n}(r) = \frac{1}{n!} \int_P \gamma \wedge (\pi r^2 + F)^n, \tag{7.18}$$

where γ is the total Chern form and F the Kähler form of P.

Proof. For an arbitrary $2q$-dimensional submanifold P in $\mathbb{C}^n$ with compact closure Weyl's Tube Formula (1.1) simplifies to

$$V_P^{\mathbb{C}^n}(r) = \sum_{c=0}^{q} \frac{k_{2c}(P)(\pi r^2)^{n-q+c}}{(2\pi)^c (n-q+c)!}. \tag{7.19}$$

Assume that P is complex and write

$$\int_P \gamma_c \wedge F^{q-c} = (\gamma_c \wedge F^{q-c})[P].$$

Then from (7.15) and (7.19) we calculate

$$\begin{aligned} V_P^{\mathbb{C}^n}(r) &= \sum_{c=0}^{q} \frac{(\gamma_c \wedge F^{q-c})[P](\pi r^2)^{n-q+c}}{(n-q+c)!(q-c)!} \\ &= \sum_{c=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\gamma_c \wedge F^k)[P](\pi r^2)^{n-k}}{(n-k)!k!} \\ &= \left\{ \gamma \wedge \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} F^k (\pi r^2)^{n-k} \right\} [P]. \end{aligned} \tag{7.20}$$

Now (7.18) follows from (7.20) and the binomial theorem. □

That the volume of a tube about a complex submanifold P of $\mathbb{C}^n$ is expressible in terms of the Chern forms was observed by Griffiths [Gs]. Formula (7.18) was proved in [Gr10].

7.3 Tubes about Complex Submanifolds of a Space $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ of Constant Holomorphic Sectional Curvature

Our goal in this section is to derive a formula that is a simultaneous generalization of the formula for the volume of a geodesic ball in $\mathbb{C}P^n(\lambda)$ and Weyl's Tube

Formula (1.1). However, we can prove a more general formula with little extra effort. Let $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ denote a Kähler manifold with constant holomorphic sectional curvature 4λ, where λ may be positive, negative or zero (see Section 6.2). We shall give a formula for the volume of a tube about a complex submanifold P of $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ in terms of the $k_{2c}(R^P - R^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)})$'s.

We shall be writing expressions in terms of trigonometric functions, but, just as in Chapters 3–6, these expressions will also make sense when λ is negative or zero. In all cases it can be checked that as $\lambda \longrightarrow 0$ the expressions reduce to the tube formulas already obtained for complex Euclidean space.

First, we derive the formula for the infinitesimal change of volume function of a complex submanifold of $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$. It is only a little more complicated than the corresponding formula for a sphere. Recall that $\mathcal{O}_P$ is the set of points in M that can be joined to P by a unique shortest geodesic meeting P perpendicularly (see the definition (2.2)), and that T_u is the Weingarten map (see the definition on page 35).

Lemma 7.8. *Let P be a topologically embedded complex submanifold of a space $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ of constant holomorphic sectional curvature 4λ. Then the infinitesimal volume function $\vartheta_u(t)$ of $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ at $p \in P$ is given by*

$$\vartheta_u(t) = \left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{2n-2q-1} \left(\cos(t\sqrt{\lambda})\right)^{2q+1} \det\left(I - \frac{\tan(t\sqrt{\lambda})}{\sqrt{\lambda}} T_u\right), \tag{7.21}$$

for $(p, tu) \in \mathcal{O}_P$ with $\|u\| = 1$.

Proof. Let ξ be a unit-speed geodesic in $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ with $\xi(0) = p$ and $\xi'(0) = u$. Because P is a complex submanifold of $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$, it is compatible with $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ by Lemma 6.16, page 97. Hence there exists a holomorphic orthonormal frame $\{e_1, \dots, Je_n\}$ such that $e_{q+1} = u$ and $\{e_1, \dots, Je_q\}$ is a basis of P_p which diagonalizes the symmetric linear transformation $R_u\colon \mathbb{K}^n_{\mathrm{hol}}(\lambda)_p \longrightarrow \mathbb{K}^n_{\mathrm{hol}}(\lambda)_p$. Since $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ is a locally symmetric space, we can choose a parallel holomorphic orthonormal frame field $t \longmapsto \{E_1(t), \dots, JE_n(t)\}$ along ξ which coincides with $\{e_1, \dots, Je_n\}$ at p. Then Theorem 6.14 implies that each $E_\alpha(t)$ is an eigenvector of both $R(t)$ and $S(t)$, where $S(t)$ is the shape operator of the tubular hypersurface P_t. Just as in the proof of Lemma 6.18, the principal curvature functions of the tubular hypersurfaces about P satisfy Riccati differential equations; however, some of the initial conditions are different.

More precisely, we have:

$$\begin{cases} \kappa'_a = \kappa_a^2 + \lambda & (a = 1, \dots, q^*), \\ \kappa'_{(q+1)^*} = \kappa^2_{(q+1)^*} + 4\lambda, & \\ \kappa'_i = \kappa_i^2 + \lambda & (i = q+2, \dots, n^*). \end{cases} \tag{7.22}$$

The initial conditions are:

$$\begin{cases} \kappa_a(0) & \text{finite} \quad \text{for } a = 1, \dots, q^*, \\ \kappa_i(0) & = -\infty \quad \text{for } i = (q+1)^*, \dots, n^*. \end{cases} \tag{7.23}$$

The solutions to (7.22) with the initial conditions (7.23) are found to be

$$\begin{cases} \kappa_a(t) &= -\dfrac{d}{dt} \log \left(\cos(t\sqrt{\lambda}) - \dfrac{\kappa_a(0)}{\sqrt{\lambda}} \sin(t\sqrt{\lambda}) \right), \\ \kappa_{(q+1)^*}(t) &= \dfrac{-2\sqrt{\lambda}}{\tan(2t\sqrt{\lambda})} = -\dfrac{d}{dt} \log\big(\sin(2t\sqrt{\lambda})\big), \\ \kappa_i(t) &= -\dfrac{d}{dt} \log\big(\sin(t\sqrt{\lambda})\big), \end{cases} \tag{7.24}$$

for $a = 1, \dots, q^*$ and $i = (q+2)^*, \dots, n^*$. When we sum the principal curvature functions given by (7.24), we get

$$\begin{aligned} \operatorname{tr} S(t) = -\frac{d}{dt} \Bigg\{ \sum_{a=1}^{q^*} \log \left(\cos(t\sqrt{\lambda}) - \frac{\kappa_a(0)}{\sqrt{\lambda}} \sin(t\sqrt{\lambda}) \right) \\ + 2(n-q-1) \log\big(\sin(t\sqrt{\lambda})\big) + \log\big(\sin(2t\sqrt{\lambda})\big) \Bigg\}. \end{aligned} \tag{7.25}$$

We substitute (7.25) into (3.14), and then integrate making use of the fact that $\vartheta_u(0) = 1$. After taking exponentials of the resulting equation, we get

$$\vartheta_u(t) = \left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}} \right)^{2n-2q-1} \cos(t\sqrt{\lambda}) \prod_{a=1}^{q^*} \left(\cos(t\sqrt{\lambda}) - \frac{\kappa_a(0)}{\sqrt{\lambda}} \sin(t\sqrt{\lambda}) \right). \tag{7.26}$$

Equation (7.26) is just another way of writing (7.21). □

Just as in the flat case, when we integrate the infinitesimal volume function over the unit sphere in $P_p^\perp$, the resulting expression depends only on the curvature tensors of the submanifold and the ambient manifold.

Corollary 7.9. *Let P be a complex submanifold of $\mathbb{K}^n_{\text{hol}}(\lambda)$, and suppose that $\exp_\nu \colon \{ (p,v) \in \boldsymbol{\nu} \mid \|v\| \le r \} \longrightarrow T(P,r) \subset \exp_\nu(\mathcal{O}_P)$ is a diffeomorphism. Then for $(p, tu) \in \mathcal{O}_P$, we have*

$$\int_{S^{2n-2q-1}(1)} \vartheta_u(t)\, du \tag{7.27}$$

$$= \left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}} \right)^{2n-2q-1} (\cos(t\sqrt{\lambda}))^{2q+1} \int_{S^{2n-2q-1}(1)} \det\left(I - \frac{\tan(t\sqrt{\lambda})}{\sqrt{\lambda}} T_u \right) du$$

$$= 2\pi^{n-q}\left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{2n-2q-1}(\cos(t\sqrt{\lambda}))^{2q+1}$$

$$\cdot\sum_{c=0}^{q}\frac{C^{2c}\big((R^P - R^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)})^c\big)}{c!(2c)!\,2^c(n-q+c-1)!}\left(\frac{\tan(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{2c}.$$

Proof. Theorem 4.10, with t replaced by $\tan(t\sqrt{\lambda})/\sqrt{\lambda}$, n replaced by $2n$, and q replaced by $2q$, can be used to integrate the right-hand side of (7.21) over $S^{2n-2q-1}(1)$. The result is (7.27). □

We are now ready to derive our first intrinsic formula for the volume of a tube about a complex submanifold of a space $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ of constant holomorphic sectional curvature 4λ.

Theorem 7.10. *Let P be a topologically embedded complex submanifold of a space $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ of constant holomorphic sectional curvature, and suppose that* $\exp_\nu\colon \{\,(p,v)\in \boldsymbol{\nu} \mid \|v\|\le r\,\} \longrightarrow T(P,r)$ *is a diffeomorphism. Then*

$$A_P^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}(r) = \frac{d}{dr}V_P^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}(r) \tag{7.28}$$

$$= 2\pi^{n-q}\big(\cos(r\sqrt{\lambda})\big)^{2n}\sum_{c=0}^{q}\frac{k_{2c}(R^P - R^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)})}{2^c(n-q+c-1)!}\left(\frac{\tan(r\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{2(n-q+c)-1}.$$

Proof. To get (7.28), we integrate (7.27) over P and use Lemmas 3.12, page 41, and 3.13, page 42, together with the definition (4.4), page 56. □

7.4 Tubes and Chern Forms

Thus the tube formula for a space $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ of constant holomorphic sectional curvature (in particular, for complex projective space $\mathbb{C}P^n(\lambda)$) is somewhat more complicated than the tube formula for Euclidean space, but it has basically the same form. Instead of a polynomial in r, the tube volume $V_P^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}(r)$ is a polynomial in trigonometric functions, and in place of the coefficients $k_{2c}(P)$ we must use the coefficients $k_{2c}(R^P - R^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)})$.

On the other hand, for a complex submanifold P of $\mathbb{C}^n$ we have derived the simple formula (7.18), which expresses $V_P^{\mathbb{C}^n}(r)$ in terms of the Chern forms and Kähler form of P. In Section 7.5 we shall find a similar formula for the volume of a tube about a complex submanifold P of a Kähler manifold $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ of constant holomorphic sectional curvature. For this it is necessary to express the coefficients $k_{2c}(R^P - R^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)})$ in terms of the Chern forms and Kähler form of P. We do the preliminary calculations with Chern forms in the present section.

In one respect the situation for $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ is better than that for $\mathbb{C}^n$, because there are many interesting compact complex submanifolds of $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$. However, the calculations are considerably more complicated.

Lemma 7.6 can be generalized immediately. Let us write

$$\tilde{\gamma}_c = \gamma_c(R^P \; - \; R^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)})$$

for the c^{th} Chern form of the curvature tensor $R^P - R^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}$. (Here, the definition of $\tilde{\gamma}_c$ is given by (6.4), page 88, where one uses the complex curvature forms of $R^P - R^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}$ in place of those of R^P.)

Lemma 7.11. *For a complex submanifold P of $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ we have*

$$(2\pi)^c(\tilde{\gamma}_c \wedge F^{q-c})(e_1, \ldots, e_{2q}) = \frac{(q-c)!}{c!\,(2c)!} C^{2c}\left((R^P - R^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)})^c\right), \qquad (7.29)$$

$$k_{2c}(R^P - R^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}) = \frac{(2\pi)^c}{(q-c)!}\int_P \tilde{\gamma}_c \wedge F^{q-c}. \qquad (7.30)$$

Proof. The calculations are exactly the same as those of Lemma 7.6 with γ_c replaced by $\tilde{\gamma}_c$. □

Now we face the more difficult problem of expressing the coefficients $k_{2c}(R^P - R^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)})$, not in terms of the $\tilde{\gamma}_c$'s, but in terms of the γ_c's. To achieve this, we need algebraic relations between the $\tilde{\gamma}_c$'s and the γ_c's. To find these algebraic relations, we must resort to some calculations with curvature forms. For this we use the structure equation version of the first Bianchi identity, which we now explain. First we do the real structure equations and afterwards the complex structure equations.

Let P be a Riemannian manifold of dimension n, and let $\{E_1, \ldots, E_n\}$ be a local orthonormal frame field on P. For $X \in \mathfrak{X}(P)$ we put

$$\omega_{ij}(X) = \left\langle \nabla_X E_i, E_j \right\rangle;$$

then $\{\, \omega_{ij} \mid i < j \,\}$ are called the **connection forms** of P relative to the local orthonormal frame field $\{E_1, \ldots, E_n\}$. Recall (see page 79) that $\{\theta_1, \ldots, \theta_n\}$ are the dual 1-forms corresponding to $\{E_1, \ldots, E_n\}$.

Lemma 7.12. *For a Riemannian manifold P and a local orthonormal frame field on P we have the* **real structure equations**

$$\left\{\begin{aligned} d\theta_i &= \sum_{j=1}^n \omega_{ij} \wedge \theta_j, \\ d\omega_{ij} &= \sum_{k=1}^n \omega_{ik} \wedge \omega_{kj} - \Omega_{ij}, \\ \omega_{ij} + \omega_{ji} &= 0, \end{aligned}\right. \qquad (7.31)$$

for $i, j = 1, \ldots, n$.

Proof. Equations (7.31) can be established directly from the definitions of the covariant derivative and the exterior differential. Let $\{E_1, \ldots, E_n\}$ be the local orthonormal frame field dual to $\{\theta_1, \ldots, \theta_n\}$, and let $X, Y \in \mathfrak{X}(P)$. Then for $i = 1, \ldots, n$ we have

$$d\theta_i(X,Y) = X\theta_i(Y) - Y\theta_i(X) - \theta_i([X,Y]); \tag{7.32}$$

this is just the definition of the exterior derivative of the 1-form θ_i. Using the definition of θ_i and well-known properties of the covariant derivative (namely (2.13)) and (2.14)), the right-hand side of (7.32) can be written as

$$\left\langle \nabla_X E_i, Y \right\rangle - \left\langle \nabla_Y E_i, X \right\rangle. \tag{7.33}$$

We expand (7.33) and get:

$$\begin{aligned} d\theta_i(X,Y) &= \left\langle \nabla_X E_i, Y \right\rangle - \left\langle \nabla_Y E_i, X \right\rangle \\ &= \sum_{j=1}^{n} \left(\left\langle \nabla_X E_i, E_j \right\rangle \langle E_j, Y \rangle - \left\langle \nabla_Y E_i, E_j \right\rangle \langle E_j, X \rangle \right) \\ &= \sum_{j=1}^{n} \left(\omega_{ij}(X)\theta_j(Y) - \omega_{ij}(Y)\theta_j(X) \right) \\ &= \sum_{j=1}^{n} (\omega_{ij} \wedge \theta_j)(X,Y). \end{aligned}$$

Since X and Y are arbitrary, we get the first equation of (7.31). The other equations of (7.31) are proved similarly. □

Now assume that P is a Kähler manifold of real dimension $2n$, and that $\{E_1, \ldots, E_n, JE_1, \ldots, JE_n\}$ is a local holomorphic orthonormal frame field on P. Put

$$\psi_{ab} = \omega_{ab} - \sqrt{-1}\,\omega_{ab^*} \qquad (1 \le a, b \le n). \tag{7.34}$$

The ψ_{ab}'s are called the **complex connection forms.**

For the next corollary, recall the definition of Ξ_{ab} given on page 88 and the definition of ϕ_a in 6.31.

Corollary 7.13. *For a Kähler manifold P and a local holomorphic orthonormal frame field on P we have the* **complex structure equations**

$$\begin{cases} d\phi_a = \sum_{b=1}^{n} \psi_{ab} \wedge \phi_b, \\ d\psi_{ab} = \sum_{c=1}^{n} \psi_{ac} \wedge \psi_{cb} - \Xi_{ab}, \\ \psi_{ab} + \overline{\psi}_{ba} = 0, \end{cases} \tag{7.35}$$

for $a, b = 1, \ldots, n$.

Proof. The first two equations of (7.35) are immediate consequences of (7.31) and the definitions (6.31) and (7.34). The last equation of (7.35) follows from the easily established fact that $\omega_{i^*j^*} = \omega_{ij}$ for all i and j. □

Using matrix notation, the real and complex structure equations can be written as

$$d\theta = \omega \wedge \theta, \qquad d\omega = \omega \wedge \omega - \Omega,$$

$$d\phi = \psi \wedge \phi, \qquad d\psi = \psi \wedge \psi - \Xi.$$

From these follow the Bianchi identities written in matrix form:

Lemma 7.14. *We have*

$$\left.\begin{aligned} \Omega \wedge \theta &= 0 \\ d\Omega - \omega \wedge \Omega + \Omega \wedge \omega &= 0 \end{aligned}\right\} \quad (\textit{Real Bianchi identities}),$$

$$\left.\begin{aligned} \Xi \wedge \phi &= 0 \\ d\Xi - \psi \wedge \Xi + \Xi \wedge \psi &= 0 \end{aligned}\right\} \quad (\textit{Complex Bianchi identities}).$$

We shall need only the first complex Bianchi identity, namely

$$\Xi \wedge \phi = 0, \tag{7.36}$$

to get the relations between the $\tilde{\gamma}_c$'s and the γ_c's that we require.

Lemma 7.15. *The $\tilde{\gamma}_c$'s are related to the γ_c's via the formula*

$$\tilde{\gamma}_c = \sum_{a=0}^{q} \binom{q-a+1}{c-a} \left(-\frac{\lambda}{\pi} F\right)^{c-a} \wedge \gamma_a, \tag{7.37}$$

or more concisely,

$$\tilde{\gamma} = \sum_{a=0}^{q} \left(1 - \frac{\lambda}{\pi} F\right)^{q-a+1} \wedge \gamma_a. \tag{7.38}$$

Proof. We use (6.4) with γ replaced by $\tilde{\gamma}$. This means that instead of using the complex curvature forms Ξ_{ab} of R^P, we must use those of $R^P - R^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}$, which we denote by $\tilde{\Xi}_{ab}$. It follows from equation (6.35), page 103, that the formula for $\tilde{\Xi}_{ab}$ in terms of Ξ_{ab} is

$$\tilde{\Xi}_{ab} = \Xi_{ab} - \lambda(\phi_a \wedge \bar{\phi}_b - 2\delta_{ab}\sqrt{-1}\, F). \tag{7.39}$$

Then from (6.4) and (7.39) we get

$$\tilde{\gamma} = \det\left(\left(1 - \frac{\lambda}{\pi} F\right)\delta_{ab} + \frac{\sqrt{-1}}{2\pi}\left(\Xi_{ab} - \lambda\phi_a \wedge \bar{\phi}_b\right)\right). \tag{7.40}$$

We expand (7.40) in the same way that we calculated the Chern forms of a space of constant holomorphic sectional curvature $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ in Theorem 6.24, page 104. The expansion of the determinant on the right-hand side of (7.40) by minors yields

$$\tilde{\gamma} = \sum_{c=0}^{q} \left(1 - \frac{\lambda}{\pi}F\right)^{q-c} \left(\frac{\sqrt{-1}}{2\pi}\right)^{c} \zeta_c(\lambda), \tag{7.41}$$

where

$$c!\,\zeta_c(\lambda) = \sum_{a_1\ldots a_c=1}^{q} \det \begin{pmatrix} \Xi_{a_1a_1} - \lambda\phi_{a_1} \wedge \bar{\phi}_{a_1} & \ldots & \Xi_{a_1a_c} - \lambda\phi_{a_1} \wedge \bar{\phi}_{a_c} \\ \vdots & \ddots & \vdots \\ \Xi_{a_ca_1} - \lambda\phi_{a_c} \wedge \bar{\phi}_{a_1} & \ldots & \Xi_{a_ca_c} - \lambda\phi_{a_c} \wedge \bar{\phi}_{a_c} \end{pmatrix}. \tag{7.42}$$

The right-hand side of (7.42) can be expanded as a polynomial in λ; thus:

$$\begin{aligned} c!\,\zeta_c(\lambda) = & \sum_{a_1\ldots a_c=1}^{q} \sum_{\sigma\in\mathfrak{S}_c} \varepsilon_\sigma \Big\{ \Xi_{a_1a_{\sigma(1)}} \wedge \ldots \wedge \Xi_{a_ca_{\sigma(c)}} \\ & - \lambda \sum_{b=1}^{c} \Xi_{a_1a_{\sigma(1)}} \wedge \ldots \wedge \phi_{a_b} \wedge \bar{\phi}_{a_{\sigma(b)}} \wedge \ldots \wedge \Xi_{a_ca_{\sigma(c)}} \\ & + \cdots + (-\lambda)^c \phi_{a_1} \wedge \bar{\phi}_{a_{\sigma(1)}} \wedge \ldots \wedge \phi_{a_c} \wedge \bar{\phi}_{a_{\sigma(c)}} \Big\}. \end{aligned} \tag{7.43}$$

Using the first Bianchi identity in the form (7.36) together with Lemma 6.21, page 102, we can rewrite (7.43) as

$$\begin{aligned} c!\,\zeta_c(\lambda) = \sum_{a_1\ldots a_c=1}^{q} & \left\{ \det \begin{pmatrix} \Xi_{a_1a_1} & \ldots & \Xi_{a_1a_c} \\ \vdots & \ddots & \vdots \\ \Xi_{a_ca_1} & \ldots & \Xi_{a_ca_c} \end{pmatrix} \right. \\ & -c\lambda\, \phi_{a_c} \wedge \bar{\phi}_{a_c} \det \begin{pmatrix} \Xi_{a_1a_1} & \ldots & \Xi_{a_1a_{c-1}} \\ \vdots & \ddots & \vdots \\ \Xi_{a_{c-1}a_1} & \ldots & \Xi_{a_{c-1}a_{c-1}} \end{pmatrix} \\ & \left. + \cdots + (-\lambda)^c \det \begin{pmatrix} \phi_{a_1} \wedge \bar{\phi}_{a_1} & \ldots & \phi_{a_1} \wedge \bar{\phi}_{a_c} \\ \vdots & \ddots & \vdots \\ \phi_{a_c} \wedge \bar{\phi}_{a_1} & \ldots & \phi_{a_c} \wedge \bar{\phi}_{a_c} \end{pmatrix} \right\}. \end{aligned} \tag{7.44}$$

From (7.44) and (6.33), page 102, it follows that

$$\zeta_c(\lambda) = \sum_{b=0}^{c} \left(2\lambda\sqrt{-1}\,F\right)^{c-b} \zeta_b(0). \tag{7.45}$$

On the other hand, from the definition of the Chern forms we have

$$\gamma_b = \left(\frac{\sqrt{-1}}{2\pi}\right)^{b} \zeta_b(0). \tag{7.46}$$

We use (7.46) to rewrite (7.45) as

$$\left(\frac{\sqrt{-1}}{2\pi}\right)^{c} \zeta_c(\lambda) = \sum_{b=0}^{c} \left(-\frac{\lambda}{\pi}\right)^{c-b} F^{c-b} \wedge \gamma_b. \tag{7.47}$$

Substituting (7.47) into (7.41) and changing the order of summation, we find that

$$\tilde{\gamma} = \sum_{b=0}^{q} \sum_{c=b}^{q} \left(1 - \frac{\lambda}{\pi}F\right)^{q-c} \wedge \left(-\frac{\lambda}{\pi}F\right)^{c-b} \wedge \gamma_b. \tag{7.48}$$

We can use the formula for the partial sums of a geometric series to simplify the right-hand side of (7.48). Thus

$$\begin{aligned} \sum_{c=b}^{q} \left(1 - \frac{\lambda}{\pi}F\right)^{q-c} \left(-\frac{\lambda}{\pi}F\right)^{c-b} &= \frac{\left(1 - \frac{\lambda}{\pi}F\right)^{q-b+1} - \left(-\frac{\lambda}{\pi}F\right)^{q-b+1}}{\left(1 - \frac{\lambda}{\pi}F\right) - \left(-\frac{\lambda}{\pi}F\right)} \\ &= \left(1 - \frac{\lambda}{\pi}F\right)^{q-b+1} - \left(-\frac{\lambda}{\pi}F\right)^{q-b+1}. \end{aligned} \tag{7.49}$$

Note that $F^{q-b+1} \wedge \gamma_b = 0$ for all b, because it is a $(2q+2)$-form on a $2q$-dimensional manifold. This observation together with (7.48) and (7.49) yields (7.38). □

Now let R be a tensor field having the same symmetries (in particular, the Kähler identity (6.1) as the curvature of a $2q$-dimensional Kähler manifold, and let $\gamma(R) = 1 + \gamma_1(R) + \cdots + \gamma_q(R)$ be its total Chern form. Actually, it will be more convenient to use the **Chern polynomial** $\gamma(R)(t)$ of R; it is defined by

$$\gamma(R)(t) = 1 + t\gamma_1(R) + \cdots + t^q\gamma_q(R). \tag{7.50}$$

We can formally factor $\gamma(R)(t)$ as[1]

$$\gamma(R)(t) = \prod_{a=1}^{q}(1 + tx_a).$$

We define a new polynomial $\hat{\gamma}(R)(t)$ by

$$\hat{\gamma}(R)(t) = \prod_{a=1}^{q}\left(1 - \frac{\lambda}{\pi}F + tx_a\right).$$

It is easy to see that formally

$$\hat{\gamma}(R)(t) = \left(1 - \frac{\lambda}{\pi}F\right)^q \gamma(R)\left(\frac{t}{1 - \dfrac{\lambda}{\pi}F}\right). \tag{7.51}$$

Next we use (7.50) and (7.51) to get a formal factorization of the total Chern form $\gamma(R^P - R^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)})$.

Lemma 7.16. *If P is a complex submanifold of $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ of complex dimension q, then*

$$\begin{aligned}\frac{\gamma(R^P - R^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)})}{1 - \dfrac{\lambda}{\pi}F} &= \prod_{a=1}^{q}\left(1 - \frac{\lambda}{\pi}F + x_a\right) \qquad (7.52)\\ &= \left(1 - \frac{\lambda}{\pi}F\right)^q \gamma(R^P)\left(\frac{1}{1 - \dfrac{\lambda}{\pi}F}\right).\end{aligned}$$

Proof. By Lemma 7.15 we have

$$\begin{aligned}\frac{\gamma(R^P - R^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)})}{1 - \dfrac{\lambda}{\pi}F} &= \sum_{a=0}^{q}\left(1 - \frac{\lambda}{\pi}F\right)^{q-a} \wedge \gamma_a(R^P)\\ &= \left(1 - \frac{\lambda}{\pi}F\right)^q + \left(1 - \frac{\lambda}{\pi}F\right)^{q-1}(x_1 + \cdots + x_q) + \cdots + x_1 \cdots x_q\\ &= \prod_{a=1}^{q}\left(1 - \frac{\lambda}{\pi}F + x_a\right).\end{aligned}$$

□

[1]The x_a are nonhomogeneous differential forms that can be quite complicated. This can be seen if one tries to factor the Chern polynomial of $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$, namely

$$\gamma(R^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)})(t) = \left(1 + t\frac{\lambda}{\pi}F\right)^{n+1} = \sum_{k=0}^{n}\binom{n+1}{k}\left(t\frac{\lambda}{\pi}F\right)^k.$$

Fortunately, explicit knowledge of the x_a's will not be necessary for the Projective Weyl Tube Formula that we shall prove in the next section.

7.5 The Projective Weyl Tube Formula

As might be expected, the Projective Weyl Tube Formula turns out to be more complicated than the Complex Weyl Tube Formula. We studied formal factorization of Chern forms in the previous section, because in its simplest form the Projective Weyl Tube Formula needs this formalism.

Before proving the Projective Weyl Tube Formula, we need another formal lemma about Chern forms. We do it for a general R, and then specialize to $R^P - R^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}$.

Lemma 7.17. *Let $\gamma(R) = 1 + \gamma_1(R) + \cdots + \gamma_q(R)$ be the total Chern form of a curvature tensor R, such that the Kähler identity* (6.1) *holds for R. Then*

$$\int_0^r \frac{2\,\pi\sin(t\sqrt{\lambda})(\cos(t\sqrt{\lambda}))^{2n-1}}{\sqrt{\lambda}} \cdot \sum_{c=0}^{q} \frac{\langle \gamma_c(R)\wedge F^{q-c}, F^q\rangle}{(q-c)!(n-q+c-1)!\,q!}\left(\frac{\pi\tan^2(t\sqrt{\lambda})}{\lambda}\right)^{n-q+c-1} dt \tag{7.53}$$

$$= \frac{1}{n!}\left\langle \frac{1}{q!}F^q, \gamma(R)\wedge \frac{\left(\frac{\pi}{\lambda}\sin^2(r\sqrt{\lambda}) + \cos^2(r\sqrt{\lambda})F\right)^n}{1-\frac{\lambda}{\pi}F}\right\rangle.$$

Proof. First, we need a formal calculation:

$$\sum_{c=0}^{q} \frac{\langle \gamma_c(R)\wedge F^{q-c}, F^q\rangle}{(q-c)!(n-q+c-1)!\,q!}\left(\frac{\pi\tan^2(t\sqrt{\lambda})}{\lambda}\right)^{n-q+c-1}$$

$$= \frac{1}{(n-1)!}\left\langle \sum_{c=0}^{q}\gamma_c(R)\wedge\sum_{b=0}^{n-1}\binom{n-1}{b}\left(\frac{\pi\tan^2(t\sqrt{\lambda})}{\lambda}\right)^{n-b-1} F^b, \frac{1}{q!}F^q\right\rangle$$

$$= \frac{1}{(n-1)!}\left\langle \gamma(R)\wedge\left(\frac{\pi\tan^2(t\sqrt{\lambda})}{\lambda}+F\right)^{n-1}, \frac{1}{q!}F^q\right\rangle.$$

Hence

$$\frac{2\,\pi\sin(t\sqrt{\lambda})(\cos(t\sqrt{\lambda}))^{2n-1}}{\sqrt{\lambda}}\sum_{c=0}^{q} \frac{\langle \gamma_c(R)\wedge F^{q-c}, F^q\rangle}{(q-c)!(n-q+c-1)!\,q!}\cdot\left(\frac{\pi\tan^2(t\sqrt{\lambda})}{\lambda}\right)^{n-q+c-1}$$

$$= \frac{2\,\pi\sin(t\sqrt{\lambda})\cos(t\sqrt{\lambda})}{(n-1)!\sqrt{\lambda}}\left\langle \gamma(R)\wedge\left(\left(\frac{\pi}{\lambda}-F\right)\sin^2(t\sqrt{\lambda})+F\right)^{n-1}, \frac{1}{q!}F^q\right\rangle.$$

When this formula is integrated with respect to t from 0 to r, the result is (7.53). $\square$

Now we can get a simple formula for the integral of the infinitesimal change of volume function over the unit sphere in $P_p^{\perp}$ in terms of the Chern forms of $R^P - R^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}$ and the Kähler form F.

Theorem 7.18. *Let P be a topologically embedded complex submanifold of $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$, and suppose that $\exp_\nu\colon \{\,(p,v) \in \boldsymbol{\nu} \mid \|v\| \le r\,\} \longrightarrow T(P,r)$ is a diffeomorphism. Then for $(p,tu) \in \mathcal{O}_P$, we have*

$$\int_0^r \int_{S^{2n-2q-1}(1)} t^{2n-2q-1}\vartheta_u(t)\,du\,dt \tag{7.54}$$

$$= \frac{1}{n!}\left\langle \frac{1}{q!}F^q,\, \gamma(R^P - R^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}) \wedge \frac{\left(\dfrac{\pi}{\lambda}\sin^2(t\sqrt{\lambda}) + \cos^2(t\sqrt{\lambda})F\right)^n}{1 - \dfrac{\lambda}{\pi}F}\right\rangle.$$

Proof. First of all, Corollary 7.9 implies that

$$t^{2n-2q-1}\int_{S^{2n-2q-1}(1)} \vartheta_u(t)\,du \tag{7.55}$$

$$= 2\pi^{n-q}\left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{2n-2q-1} (\cos(t\sqrt{\lambda}))^{2q+1}$$

$$\cdot \sum_{c=0}^{q} \frac{C^{2c}\left((R^P - R^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)})^c\right)}{c!(2c)!\,2^c(n-q+c-1)!}\left(\frac{\tan(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{2c}.$$

Next we use (7.29) to rewrite the right-hand side of (7.55) as

$$\frac{2\,\pi\sin(t\sqrt{\lambda})(\cos(t\sqrt{\lambda}))^{2n-1}}{\sqrt{\lambda}} \sum_{c=0}^{q} \frac{\left\langle \gamma_c(R^P - R^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}) \wedge F^{q-c}, F^q\right\rangle}{(q-c)!(n-q+c-1)!\,q!} \tag{7.56}$$

$$\cdot\left(\frac{\pi\tan^2(t\sqrt{\lambda})}{\lambda}\right)^{n-q+c-1}.$$

To compute the integral from 0 to r of (7.56), we use (7.53) with R replaced by $R^P - R^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}$. Thus (7.54) follows from (7.55), (7.56) and (7.53). □

Moreover, we obtain the formula for the tube volume $V_P^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}(r)$ in terms of the Kähler form and Chern forms of $R^P - R^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}$:

Theorem 7.19. *Let P be a complex submanifold of $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$, and suppose that $\exp_\nu\colon \{ (p,v) \in \boldsymbol{\nu} \mid \|v\| \le r \} \longrightarrow T(P,r)$ is a diffeomorphism. Then*

$$V_P^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}(r) = \frac{1}{n!}\int_P \gamma(R^P - R^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}) \wedge \frac{\left(\dfrac{\pi}{\lambda}\sin^2(r\sqrt{\lambda}) + \cos^2(r\sqrt{\lambda})F\right)^n}{1 - \dfrac{\lambda}{\pi}F}. \tag{7.57}$$

Proof. When we integrate (7.54) over P and use Lemma 3.13, we get (7.57). □

Finally, we derive an expression for $V_P^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}(r)$ in terms of the Chern forms of P:

Theorem 7.20. (The Projective Weyl Tube Formula.) *Let P be a complex submanifold of $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$, and suppose that*

$$\exp_\nu\colon \{ (p,v) \in \boldsymbol{\nu} \mid \|v\| \le r \} \longrightarrow T(P,r)$$

is a diffeomorphism. Then

$$V_P^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}(r) = \frac{1}{n!}\int_P \prod_{a=1}^{q}\left(1 - \frac{\lambda}{\pi}F + x_a\right) \wedge \left(\frac{\pi}{\lambda}\sin^2(r\sqrt{\lambda}) + \cos^2(r\sqrt{\lambda})F\right)^n. \tag{7.58}$$

Proof. Formula (7.58) is immediate from Lemma 7.16 and Theorem 7.19. □

7.6 Tubes about Complex Hypersurfaces of Complex Projective Space $\mathbb{C}P^{n+1}(\lambda)$

In this section we combine the formulas of Chapter 6 for the Chern classes of a complex hypersurface $M_d(\lambda)$ of $\mathbb{C}P^{n+1}(\lambda)$ with formulas (7.51) and (7.58). In this way we get an especially simple formula for the volume of a tube about $M_d(\lambda)$.

Lemma 7.21. *For a complex hypersurface $M_d(\lambda) \subset \mathbb{C}P^{n+1}(\lambda)$, we have*

$$\left[\gamma\left(R^{M_d(\lambda)} - R^{\mathbb{C}P^{n+1}(\lambda)}\right)\right] = \left[\frac{1}{1 + \dfrac{(d-1)\lambda}{\pi}F}\right]. \tag{7.59}$$

Proof. Theorem 6.35 implies that

$$[\gamma(M_d(\lambda))(t)] = [\gamma(R^{M_d(\lambda)})] = \left[\frac{\left(1 + t\dfrac{\lambda}{\pi}F\right)^{n+2}}{1 + t\dfrac{\lambda d}{\pi}F}\right]. \tag{7.60}$$

From (7.52) we have

$$\gamma(R^{M_d(\lambda)} - R^{\mathbb{C}P^{n+1}(\lambda)}) = \left(1 - \frac{\lambda}{\pi}F\right)^{n+1} \gamma(R^{M_d(\lambda)})\left(\frac{1}{1 - \dfrac{\lambda}{\pi}F}\right). \tag{7.61}$$

We take $t = \left(1 - \dfrac{\lambda}{\pi}F\right)^{-1}$ in (7.60) and substitute the resulting expression into the right-hand side of (7.61). Thus

$$\left(1 + t\frac{\lambda}{\pi}F\right) \qquad \text{becomes} \qquad \left(1 - \frac{\lambda}{\pi}F\right)^{-1},$$

and

$$\left(1 + td\frac{\lambda}{\pi}F\right) \qquad \text{becomes} \qquad \frac{1 - \dfrac{(d-1)\lambda}{\pi}F}{1 - \dfrac{\lambda}{\pi}F}.$$

After some calculation we obtain (7.59). □

Next we show that the volume of a tube about a complex hypersurface $M_d(\lambda)$ of $\mathbb{C}P^{n+1}(\lambda)$ depends only on λ, r and the degree d of the hypersurface.

Theorem 7.22. *Let $M_d(\lambda)$ be a complex hypersurface of degree d in complex projective space $\mathbb{C}P^{n+1}(\lambda)$, and suppose that $\exp_\nu\colon \{ (p,v) \in \boldsymbol{\nu} \mid \|v\| \le r \} \longrightarrow T(M_d(\lambda), r)$ is a diffeomorphism. Then the volume of a tube of radius r about $M_d(\lambda)$ is given by*

$$V_{M_d(\lambda)}^{\mathbb{C}P^{n+1}(\lambda)}(r) = \frac{1}{(n+1)!}\left(\frac{\pi}{\lambda}\right)^{n+1}\left(1 - (1 - d\sin^2(r\sqrt{\lambda}))^{n+1}\right). \tag{7.62}$$

Proof. We have by Theorem 7.19 and Lemma 7.21 that

$$V_{M_d(\lambda)}^{\mathbb{C}P^{n+1}(\lambda)}(r) = \frac{1}{(n+1)!}\int_{M_d(\lambda)} \frac{\left(\dfrac{\pi}{\lambda}\sin^2(r\sqrt{\lambda}) + \cos^2(r\sqrt{\lambda})F\right)^{n+1}}{\left(1 - \dfrac{\lambda}{\pi}F\right)\left(1 + (d-1)\dfrac{\lambda}{\pi}F\right)}. \tag{7.63}$$

The power series expansion of the denominator of the integrand on the right-hand side of (7.63) is easily computed. We have

$$\frac{1}{\left(1-\dfrac{\lambda}{\pi}F\right)\left(1+(d-1)\dfrac{\lambda}{\pi}F\right)} \tag{7.64}$$

$$=\sum_{c=0}^{\infty}\frac{1}{d}\left(1-(1-d)^{c+1}\right)\left(\frac{\lambda}{\pi}F\right)^{c}.$$

Therefore, the integrand on the right-hand side of (7.63) can be written as

$$\frac{\left(\frac{\pi}{\lambda}\sin^2(r\sqrt{\lambda})+\cos^2(r\sqrt{\lambda})F\right)^{n+1}}{\left(1-\dfrac{\lambda}{\pi}F\right)\left(1+(d-1)\dfrac{\lambda}{\pi}F\right)}=\left\{\sum_{c=0}^{\infty}\frac{\left(1-(1-d)^{c+1}\right)}{d}\left(\frac{\lambda}{\pi}F\right)^{c}\right\} \tag{7.65}$$

$$\cdot\left\{\sum_{a=0}^{n+1}\binom{n+1}{a}\left(\frac{\pi\sin^2(r\sqrt{\lambda})}{\lambda}\right)^{n+1-a}\left(\cos^2(r\sqrt{\lambda})F\right)^{a}\right\}.$$

We are interested only in the component of degree n on the right-hand side of (7.65); it is given by

$$\frac{\pi}{d\lambda}\sum_{a=0}^{n+1}\binom{n+1}{a}\left(1-(1-d)^{n+1-a}\right)\left(\sin^2(r\sqrt{\lambda})\right)^{n+1-a}\left(\cos^2(r\sqrt{\lambda})\right)^{a}F^n \tag{7.66}$$

$$=\frac{\pi}{d\lambda}\left\{1-\sum_{a=0}^{n+1}\binom{n+1}{a}\left((1-d)\sin^2(r\sqrt{\lambda})\right)^{n+1-a}\left(\cos^2(r\sqrt{\lambda})\right)^{a}\right\}F^n$$

$$=\left\{1-\left((1-d)\sin^2(r\sqrt{\lambda})+\cos^2(r\sqrt{\lambda})\right)^{n+1}\right\}\frac{\pi}{d\lambda}F^n$$

$$=\left\{1-\left(1-d\sin^2(r\sqrt{\lambda})\right)^{n+1}\right\}\frac{\pi}{d\lambda}F^n.$$

Now (7.62) follows from (7.65) and (7.66), when we integrate (7.66) over $M_d(\lambda)$. □

Corollary 7.23. *The volume $V_{M_d(\lambda)}^{\mathbb{C}P^{n+1}(\lambda)}(r)$ of a tube about a complex hypersurface $M_d(\lambda)$ in $\mathbb{C}P^{n+1}(\lambda)$ depends only on λ and the degree d of the hypersurface.*

7.7 Kähler Deformations

We now show that the volume of a tube about a Kähler submanifold of a space of constant holomorphic sectional curvature is invariant under Kähler deformations. Let $d = d' + d''$ be the usual decomposition of the exterior differential into its $(1,0)$ and $(0,1)$ components.

Definition. *Let M be a Kähler manifold with Kähler form F. A* **Kähler deformation** *of M consists of a change of Kähler form*

$$F \longmapsto F + \sqrt{-1}\, d'd''f,$$

where f is a real-valued differentiable function on M.

It is easy to prove that the deformed Kähler form is also a Kähler form.

Theorem 7.24. *Let P be a compact Kähler manifold with Kähler form F. Define $V_P^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}(r)$ by (7.58). Then $V_P^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}(r)$ is the same for all Kähler deformations.*

Proof. The proof is elementary and is essentially the same as that of [BGM, page 117]. We do only the case $\lambda = 0$. Let $F \longmapsto F + \sqrt{-1}\, d'd''f$ be a Kähler deformation. The metric also changes, but the total Chern form γ becomes $\gamma + d\alpha$. Then

$$(\gamma + d\alpha) \wedge (\pi r^2 + F + \sqrt{-1}\, d'd''f)^n = \gamma \wedge (\pi r^2 + F)^n + d\eta$$

for some (nonhomogeneous) differential form η. Thus by Stokes' Theorem

$$V_P^{\mathbb{C}^n}(r) = \frac{1}{n!}\int_P (\gamma + d\alpha) \wedge (\pi r^2 + F + \sqrt{-1}\, d'd''f)^n,$$

and the theorem follows. □

Theorem 7.24 is related to the notion of generalized Chern numbers.

Definition. *Let $(i_1, \dots, i_{k-1})$ be a sequence of numbers such that*

$$i_1 + i_2 + \cdots + i_{k-1} + k = n.$$

Then the **generalized Chern number** *$c_{i_1} c_{i_2} \dots c_{i_{k-1}}[F]^k(M)$ is given by*

$$c_{i_1} c_{i_2} \dots c_{i_{k-1}}[F]^k(M) = \int_M \gamma_{i_1} \wedge \dots \wedge \gamma_{i_{k-1}} \wedge F^k.$$

The Chern numbers are topological invariants, but the generalized Chern numbers are not; for example, they change when the metric (or equivalently F) is multiplied by a constant. However, each generalized Chern number depends only on the cohomology class of F. The generalized Chern numbers are also invariant under Kähler deformations (see [BGM, page 117]).

7.8 Tubes about Totally Real Submanifolds of a Space $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ of Constant Holomorphic Sectional Curvature

In general the calculation of the volume of a tube about a noncomplex submanifold of $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ is a great deal more difficult than the calculation of the volume of a tube about a complex submanifold of $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$, because the Chern forms are unavailable. Indeed, in [GV4] it is shown that the volume of a tube about two isometric embeddings of the same Riemannian manifold into $\mathbb{C}P^n(\lambda)$ may differ. This happens when one of the embeddings is complex and the other is not. However, there is one case of a noncomplex embedding where the calculation of the volume of a tube is simple.

Definition. *A submanifold P of an almost complex manifold M is called* **totally real** *provided that the almost complex structure J of M maps tangent vectors to P into normal vectors.*

The most obvious example of a totally real submanifold is a totally real vector subspace of $\mathbb{C}^n$. We now describe another example. Let $\mathbb{R}P^n(\lambda)$ denote real projective space with the metric that has constant sectional curvature λ.

Lemma 7.25. *There is a natural embedding of real projective space $\mathbb{R}P^n(\lambda)$ as a totally real totally geodesic submanifold of $\mathbb{C}P^n(\lambda)$.*

Proof. Let V be a totally real subspace of $\mathbb{C}^{n+1}$ of dimension $n+1$. Without loss of generality, we can identify V with $\mathbb{R}^{n+1}$. Then the projection $\pi\colon \mathbb{C}^{n+1} \longrightarrow \mathbb{C}P^n(\lambda)$ restricts to a projection $\pi\colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}P^n(\lambda)$. Thus we get the standard embedding of $\mathbb{R}P^n(\lambda)$ in $\mathbb{C}P^n(\lambda)$ that maps homogeneous coordinates in $\mathbb{R}P^n(\lambda)$ to homogeneous coordinates in $\mathbb{C}P^n(\lambda)$. This embedding is totally geodesic. □

Lemma 7.26. *Let P be a topologically embedded n-dimensional totally real totally geodesic submanifold of a space $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ of constant holomorphic sectional curvature 4λ. Then the infinitesimal change of volume function $\vartheta_u(t)$ is given by*

$$\vartheta_u(t) = \left(\frac{\sin(2t\sqrt{\lambda})}{2t\sqrt{\lambda}}\right)^{n-1} \cos(2t\sqrt{\lambda}) \tag{7.67}$$

for $(p, tu) \in \mathcal{O}_P$ with $\|u\| = 1$.

Proof. We proceed as in the proof of Lemma 7.8, making modifications where necessary. The equations for the principal curvature functions along a geodesic

normal to P are

$$\begin{cases} \kappa_a(t) &= \sqrt{\lambda}\tan(t\sqrt{\lambda}) \\ \kappa_n(t) &= 2\sqrt{\lambda}\tan(2t\sqrt{\lambda}) \\ \kappa_i(t) &= \dfrac{-\sqrt{\lambda}}{\tan(t\sqrt{\lambda})} \end{cases} \tag{7.68}$$

for $a = 1, \dots, n-1$ and $i = n+2, \dots, 2n$. When we sum the principal curvature functions given by (7.68) and use (3.14), we get

$$\begin{aligned} \frac{\vartheta_u'(t)}{\vartheta_u(t)} &= -\frac{n-1}{t} - (n-1)\sqrt{\lambda}\tan(t\sqrt{\lambda}) \\ &\quad - 2\sqrt{\lambda}\tan(2t\sqrt{\lambda}) + (n-1)\frac{\sqrt{\lambda}}{\tan(t\sqrt{\lambda})}. \end{aligned} \tag{7.69}$$

When we integrate and exponentiate (7.69), we get (7.68)). □

Theorem 7.27. *The volume of a tube of radius r about $\mathbb{R}P^n(\lambda)$ embedded as a totally geodesic totally real submanifold of $\mathbb{C}P^n(\lambda)$ is given by*

$$V_{\mathbb{R}P^n(\lambda)}^{\mathbb{C}P^n(\lambda)}(r) = \frac{\pi^n}{\lambda^n n!}\sin(2r\sqrt{\lambda})^n. \tag{7.70}$$

Proof. Equation (7.70) is a consequence of Lemma 7.26, the identity

$$\Gamma\Big(\frac{n}{2}\Big)\Gamma\Big(\frac{1}{2}(n+1)\Big) = \frac{(n-1)!\sqrt{\pi}}{2^{n-1}},$$

and the fact that

$$\text{volume}(\mathbb{R}P^n(\lambda)) = \frac{1}{2}\,\text{volume}(S^n(\lambda)) = \frac{\pi^{\frac{1}{2}(n+1)}}{\Gamma\big(\frac{1}{2}(n+1)\big)\,\lambda^{\frac{n}{2}}}.$$ □

7.9 Problems

7.1 The covariant derivative of a double form ω of type (p,q) on a Riemannian manifold M is defined by

$$\begin{aligned} \nabla_Z(\omega)(X_1, \dots, X_p) &= \nabla_Z(\omega(X_1, \dots, X_p)) \\ &\quad - \sum_{j=1}^{p}\omega(X_1, \dots, \nabla_Z X_j, \dots, X_p) \end{aligned}$$

for $Z, X_1, \ldots, X_p \in \mathfrak{X}(M)$. Define an operator D which assigns to each double form ω of type (p,q) a double form $D\omega$ of type $(p+1,q)$ by the formula

$$(D\omega)(X_1,\ldots,X_{p+1}) = \sum_{j=1}^{p+1}(-1)^{j+1}\nabla_{X_j}(\omega)(X_1,\ldots,\hat{X}_j,\ldots,X_{p+1}).$$

for $X_1,\ldots,X_{p+1} \in \mathfrak{X}(M)$.

a. Show that for $q = 0$ the operator D reduces to the ordinary exterior derivative and hence is independent of the Riemannian metric.

b. Establish the formulas

$$\begin{aligned}\nabla_Z(\omega\wedge\theta) &= \nabla_Z(\omega)\wedge\theta + \omega\wedge\nabla_Z(\theta),\\ D(\omega\wedge\theta) &= D\omega\wedge\theta + (-1)^p\omega\wedge D\theta,\end{aligned}$$

where ω is a double form of type (p,q) and θ is an arbitrary double form.

c. Let R be the double form of type $(2,2)$ determined by the curvature operator of M. Show that for all c

$$D(R^c) = 0.$$

This is the generalization for double forms of the second Bianchi identity.

7.2 Let P be a topologically embedded complex submanifold of $\mathbb{C}^n$ with compact closure; assume that $\exp_\nu\colon \{\, (p,v) \in \boldsymbol{\nu} \mid \|v\| \le r \,\} \longrightarrow T(P,r)$ is a diffeomorphism. Show that the infinitesimal change of volume function of P is related to the total Chern form and Kähler form of P by the formula

$$\int_0^r \int_{S^{2n-1}(1)} t^{2n-1}\vartheta_u(t)\,du\,dt = \frac{1}{n!}\left\langle \gamma\wedge(\pi r^2 + F)^n, \frac{1}{q!}F^q \right\rangle. \tag{7.71}$$

(Prove (7.71) directly or take the limit as $\lambda \longrightarrow 0$ in (7.54).)

7.3 For a complete intersection $P = P_{a_1\ldots a_r}(\lambda) \subset \mathbb{C}P^{n+r}(\lambda)$ show that

$$\left[\gamma\left(R^P - R^{\mathbb{C}P^{n+r}(\lambda)}\right)\right] = \left[\frac{1}{\prod_{c=1}^{r}\left(1 + \dfrac{(a_c-1)\lambda}{\pi}F\right)}\right]. \tag{7.72}$$

7.4 Let $P = P_{a_1 \dots a_r}(\lambda) \subset \mathbb{C}P^{n+r}(\lambda)$ be a complete intersection (defined as the zeros of polynomials of degrees $a_1, \dots, a_r$). Assume that (3.3) holds. Show that

$$V_P^{\mathbb{C}P^{n+r}(\lambda)}(r) = \frac{1}{n!} \int_P \frac{\left(\frac{\pi}{\lambda} \sin^2(r\sqrt{\lambda}) + \cos^2(r\sqrt{\lambda}) F\right)^n}{\left(1 - \frac{\lambda}{\pi} F\right) \prod_{c=1}^{r} \left(1 + (a_c - 1)\frac{\lambda}{\pi} F\right)}. \tag{7.73}$$

Chapter 8

Comparison Theorems for Tube Volumes

In this chapter we shall show that there is a simultaneous generalization of Weyl's Tube Formula and the Bishop-Günther Inequalities. Imagine a manifold M equipped with a family of Riemannian metrics. Intuitively, it is clear that if one varies the metric of M so that its curvature increases, then volumes in M decrease. For geodesic balls the Bishop-Günther Inequalities (Theorem 3.17, page 45) show this clearly. The situation for tubes is more complicated and interesting, however, as we shall see.

Section 8.1 contains the definitions of focal and cut-focal points; these definitions generalize the notions of conjugate and cut points. We also show that with suitable assumptions $A_P^M(r)$ is the derivative of $V_P^M(r)$ for *all* $r \geq 0$. Eventually, this allows us to extend the validity of Weyl's Tube Formula for $\mathbb{R}^n$, but as an inequality.

In Section 8.2 we generalize Weyl's Tube Formula to a submanifold P of a complete manifold M of nonpositive or nonnegative sectional curvature. The formula changes in two ways. In the first place, it becomes an inequality. Secondly, the coefficients use the difference $R^P - R^M$ of curvature tensors instead of the curvature tensor R^P.

Section 8.3 is devoted to deriving inequalities similar to those of Section 8.2, but for the volume of a tube about a submanifold P of a manifold M whose sectional curvature satisfies $K^M \geq \lambda$ or $K^M \leq \lambda$, where λ is a constant. In Weyl's original paper there is a formula for the volume of a tube about a submanifold of a sphere. We recover this formula as a corollary. In Section 8.4 we show how lower bounds on Ricci curvature yield upper bounds on tube volumes.

Tubes about a Kähler submanifold of a Kähler manifold whose holomorphic and antiholomorphic sectional curvatures are bounded above or below are studied in Section 8.5, where we generalize and sharpen the results of Chapters 6 and 7.

Inequalities of Heintze and Karcher [HK] are discussed in Section 8.6. In Section 8.7 we prove Gromov's improvement of the Bishop-Günther Inequalities. Finally, in Section 8.8 we discuss refined comparison theorems for surfaces.

In Sections 8.2–8.7 the comparison theorems are all of the type where we compare volumes in a given Riemannian manifold with volumes in a model space (Euclidean space in Section 8.2, a space of constant sectional curvature in Section 8.3, and a space of constant holomorphic sectional curvature in Section 8.5). It is possible to dispense with the model space. We use a method that goes back to the papers of M. Bôcher [Bôcher1], [Bôcher2], [Bôcher3] in order to obtain in Section 8.8 comparison theorems for two surfaces M and $\widetilde{M}$, for which in an appropriate sense the curvature of M is larger than that of $\widetilde{M}$. In Section 8.9 we describe a method developed by J. Eschenburg and E. Heintze which applies to Riemannian Manifolds of any dimension.

Unless stated otherwise in this chapter P will be a q-dimensional submanifold with compact closure topologically embedded in a complete n-dimensional Riemannian manifold M.

8.1 Focal Points and Cut-focal Points

As in Chapter 2 the normal bundle of P in M will be denoted by $\boldsymbol{\nu}$, and $\exp_\nu$ will denote the exponential map of the normal bundle. Then $\exp_\nu$ is defined and nonsingular in a neighborhood of the zero section of $\boldsymbol{\nu}$. The first order of business is to generalize the notions of conjugate and cut points to submanifolds.

Definition. *A* **focal point** *of P is a point $m \in M$ such that the exponential map* $\exp_\nu$ *of the normal bundle $\boldsymbol{\nu}$ of $P \subset M$ is singular somewhere on* $\exp_\nu^{-1}(m)$.

Let $e_f\colon \{\, (p,u) \mid p \in P,\ u \in P_p^\perp,\ \|u\| = 1 \,\} \longrightarrow \mathbb{R}$ be the function defined by

$$e_f(p,u) = \inf\{\, t > 0 \mid \mathrm{kernel}(((\exp_\nu)_*)_{(p,tu)}) \neq 0 \,\}.$$

In words $e_f(p,u)$ is the distance from p to its first focal point along the geodesic $t \longmapsto \exp_\nu(p,tu)$.

Definition. *Let ξ be a geodesic meeting P orthogonally, and let m be a point on ξ. We say that m is a* **cut-focal point (along** ξ**)**, *provided distance from m to P is no longer minimized along ξ after m. In other words, m is the first point on ξ beyond which there is a point $\widetilde{m}$ on ξ and a geodesic $\widetilde{\xi}$ from $\widetilde{m}$ to P that meets P orthogonally such that the distance from $\widetilde{m}$ to P along $\widetilde{\xi}$ is less than the distance from $\widetilde{m}$ to P along ξ.*

There is a function e_c for cut-focal points that corresponds to e_f. Let $e_c\colon \{\, (p,u) \mid p \in P,\ u \in P_p^\perp,\ \|u\| = 1 \,\} \longrightarrow \mathbb{R}$ be defined by

$$e_c(p,u) = \sup\{\, t > 0 \mid \mathrm{distance}(\exp_\nu(p,tu), P) = t \,\}.$$

Thus $e_c(p,u)$ is the distance from p to its cut-focal point in the direction u if such a cut focal point exists; otherwise $e_c(p,u) = \infty$. On the other hand, $e_f(p,u)$ is the distance from p to its nearest focal point in the direction u. There is a fundamental inequality between $e_c(p,u)$ and $e_f(p,u)$:

Lemma 8.1. *For all $p \in P$ and $u \in P_p^{\perp}$ we have*

$$e_c(p,u) \leq e_f(p,u).$$

(For proof of Lemma 8.1 see [Am1, page 63] or [Sakai, page 96].)

The set $\mathcal{O}_P$ defined in Chapter 2 can now be written as

$$\mathcal{O}_P = \{\, (p,tu) \in \boldsymbol{\nu} \mid \|u\| = 1 \text{ and } 0 \leq t < e_c(p,u) \,\}.$$

It is obvious that the boundary of $\exp_\nu(\mathcal{O}_P)$ is the set of cut-focal points. Also, Lemma 8.1 implies that $\exp_\nu\colon \mathcal{O}_P \longrightarrow \exp_\nu(\mathcal{O}_P)$ is a diffeomorphism.

The original definitions of a tube and a tubular hypersurface that we gave in Chapter 3, namely

$$\begin{aligned} T(P,r) &= \{\, m \in M \mid \text{ there exists a geodesic } \xi \text{ of length} \\ &\qquad\qquad L(\xi) \leq r \text{ from } m \text{ meeting } P \text{ orthogonally} \,\}, \\ P_t &= \{\, m \in T(P,r) \mid \text{distance}(m,P) = t \,\}, \end{aligned}$$

make sense for any topologically embedded submanifold of any Riemannian manifold. Both $T(P,r)$ and P_t are measurable; therefore, they have volumes, which are just $V_P^M(r)$ and $A_P^M(r)$.

Until now we have been assuming that a tube $T(P,r)$ satisfies condition (3.3), which says that $\exp_\nu$ maps a tube of radius r about the zero section in $\boldsymbol{\nu}$ diffeomorphically onto $T(P,r)$. Notice that (3.3) implies that

$$T(P,r) \subseteq \exp_\nu(\mathcal{O}_P).$$

Let us consider a condition that is weaker than (3.3), namely

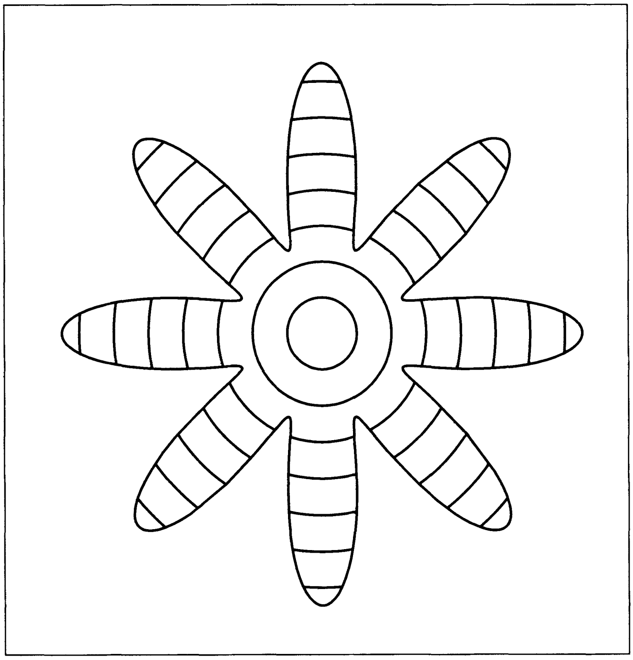
$$\exp_\nu \text{ maps a subset of } \{\, (p,v) \in \boldsymbol{\nu} \mid \|v\| \leq r \,\} \qquad (8.1)$$

$$\text{diffeomorphically onto } T(P,r).$$

In many cases we can get by with (8.1) in place of (3.3), and we consider $\exp_\nu$ defined only on this subset. For those cases when this is not possible, it will be convenient to introduce the following notion:

Definition. *The* **minimal focal distance** *of P in M is*

$$\operatorname{minfoc}(P) = \inf\{\, e_c(p,u) \mid (p,u) \in \boldsymbol{\nu},\ \|u\| = 1 \,\}.$$



The exponential map $\exp_m$ **may map a proper subset of a ball in** M_m **into a ball in** M

Notice that because we are assuming that M is complete, condition (3.3) is implied by the condition

$$0 \le r \le \operatorname{minfoc}(P). \tag{8.2}$$

In the case that P is a point m, the minimal focal distance coincides with the injectivity radius at m (as defined, for example, in [Bessel, page 130]).

Next we show that Lemmas 3.12 and 3.13 hold under assumption (8.1) instead of assumption (3.3), but in order to do so we need to extend the definition of the infinitesimal volume function. This we do in the most naive way possible by defining

$$\Theta_u(t) = \begin{cases} \vartheta_u(t) & \text{for } t \le e_c(p,u), \\ 0 & \text{for } t > e_c(p,u). \end{cases}$$

The revised version of Lemma 3.12 is as follows:

Lemma 8.2. *For all* $r \geq 0$ *we have*

$$\mathrm{A}_P^M(r) = r^{n-q-1} \int_P \int_{S^{n-q-1}(1)} \Theta_u(r)\, du\, dP. \tag{8.3}$$

Proof. When $0 \leq r \leq \operatorname{minfoc}(P)$, the inverse image $\exp_\nu^{-1}(P_r)$ of a parallel hypersurface P_r coincides with $\{ (p,v) \in \boldsymbol{\nu} \mid \|v\| = r \}$. For larger values of r, $\exp_\nu^{-1}(P_r)$ is a proper subset of $\{ (p,v) \in \boldsymbol{\nu} \mid \|v\| = r \}$, in fact, a submanifold with boundary. In spite of this complication, the proof that $*d\sigma$ is the volume element of P_r for each r is the same as that of Lemma 3.12. We can transfer the integration of $*d\sigma$ over P_r to $\exp_\nu^{-1}(P_r)$ just as in the proof of Lemma 3.12:

$$\mathrm{A}_P^M(r) = \int_{P_r} *d\sigma = \int_{\exp_\nu^{-1}(P_r)} \exp_\nu^*(*d\sigma).$$

Equation (3.12) implies that

$$\Theta_u(t) = \left\{ \omega\left(\frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n} \right) \circ \exp_\nu \right\}(p, tu)$$

for $0 \leq t \leq e_c(p,u)$. Because $\Theta_u(t)$ vanishes on the complement of the set $\exp_\nu^{-1}(T(P,r))$ in $\{ (p,v) \in \boldsymbol{\nu} \mid \|v\| = r \}$, it follows from (3.21) that for all $r \geq 0$

$$\mathrm{A}_P^M(r) = \int_{\{ (p,v) \mid v \in P_p^\perp,\, \|v\| = r \}} \Theta_u(r)\, d\boldsymbol{\nu} = \int_P \int_{S^{n-q-1}(r)} \Theta_u(r)\, du\, dP.$$

The rest of the proof (which consists in transferring the integration from $S^{n-q-1}(r)$ to $S^{n-q-1}(1)$) is the same as that of Lemma 3.12, and so we get (8.3). $\square$

Lemma 8.3. *For all $r \geq 0$ we have*

$$\mathrm{V}_P^M(r) = \int_0^r \mathrm{A}_P^M(t)\, dt = \int_0^r \int_P \int_{S^{n-q-1}(1)} t^{n-q-1} \Theta_u(t)\, du\, dP\, dt. \tag{8.4}$$

Proof. The proof is word for word that of Lemma 3.13. The only difference is that Lemma 8.2 is used in place of Lemma 3.12. $\square$

Remark. The function $r \longmapsto A_P^M(r)$ need not be continuous. See the paper of Hartman [Har] for examples and a discussion of this problem.

8.2 Tubes about Submanifolds of a Space of Nonnegative or Nonpositive Sectional Curvature

We are now ready to generalize Weyl's Tube Formula as an inequality. The ambient manifold $\mathbb{R}^n$ will be replaced by a complete Riemannian manifold M of nonnegative or nonpositive sectional curvature. As usual K^M will denote the sectional curvature of M.

The situation for nonnegative curvature is slightly better than that for nonpositive curvature:

Theorem 8.4. *Suppose* $K^M \geq 0$.

(i) *For* $0 \leq r \leq \operatorname{minfoc}(P)$ *we have*

$$\begin{aligned} V_P^M(r) &\leq \frac{(\pi r^2)^{(n-q)/2}}{(\frac{1}{2}(n-q))!} \sum_{c=0}^{[q/2]} \frac{k_{2c}(R^P - R^M) r^{2c}}{(n-q+2)(n-q+4)\cdots(n-q+2c)} \qquad (8.5) \\ &\leq \int_0^r t^{n-q-1} \int_P \int_{S^{n-q-1}(1)} \left(1 - \frac{t}{q}\langle H, u\rangle\right)^q du\, dP\, dt \\ &= \sum_{c=0}^{[q/2]} \frac{(\pi r^2)^{(n-q)/2} q!\, r^{2c}}{4^c (q-2c)!\, c!\, (\frac{1}{2}(n-q+2c))!} \int_P \left(\frac{1}{q}\|H\|\right)^{2c} dP. \end{aligned}$$

(ii) *For all* $r \geq 0$ *we have*

$$\begin{aligned} V_P^M(r) &\leq \int_0^r \int_P \int_{S^{n-q-1}(1)} t^{n-q-1} \max(\det(I - tT_u), 0)\, du\, dP\, dt \qquad (8.6) \\ &\leq \int_0^r \int_P \int_{S^{n-q-1}(1)} t^{n-q-1} \max\left(\left(1 - \frac{t}{q}\langle H, u\rangle\right)^q, 0\right) du\, dP\, dt. \end{aligned}$$

Here, the $k_{2c}(R^P - R^M)$'s that occur on the right-hand side of (8.5) are the same expressions as the $k_{2c}(P)$'s that occur in Weyl's Tube Formula (1.1). (In (1.1) we wrote $k_{2c}(P)$, but more properly we should have used the notation $k_{2c}(R^P)$.) So the first inequality of the right-hand side of (8.5) is the same as the right-hand side of (1.1), except that R^P has been replaced by $R^P - R^M$. See the discussion before Theorem 4.10 regarding the tensor $R^P - R^M$. The mean curvature vector field H is defined in Section 6.7. Inequalities (8.5) and (8.6) of course coincide for $0 \leq r \leq \operatorname{minfoc}(P)$.

There is a corresponding but weaker result for the nonpositive curvature case:

Theorem 8.5. *Suppose* $0 \leq r \leq \operatorname{minfoc}(P)$. *If* $K^M \leq 0$, *then*

$$V_P^M(r) \geq \frac{(\pi r^2)^{(n-q)/2}}{(\frac{1}{2}(n-q))!} \sum_{c=0}^{[q/2]} \frac{k_{2c}(R^P - R^M) r^{2c}}{(n-q+2)(n-q+4)\cdots(n-q+2c)}. \qquad (8.7)$$

Nice simplifications of Theorems 8.4 and 8.5 occur when the dimension of the submanifold P is small:

Corollary 8.6. *Suppose that* $0 \leq r \leq \operatorname{minfoc}(P)$, *and that* $\dim(P) \leq 3$.

(i) *If* $K^M \geq 0$, *then* $V_P^M(r) \leq V_P^{\mathbb{R}^n}(r)$.

(ii) *If* $K^M \leq 0$, *then* $V_P^M(r) \geq V_P^{\mathbb{R}^n}(r)$.

A clarification is in order. Let P be an abstract Riemannian manifold of dimension q, possibly with boundary, and let n be a positive integer. We can define $V_P^{\mathbb{R}^n}(r)$ abstractly by the formula

$$V_P^{\mathbb{R}^n}(r) = \frac{(\pi r^2)^{(n-q)/2}}{(\frac{1}{2}(n-q))!} \sum_{c=0}^{[q/2]} \frac{k_{2c}(R^P) r^{2c}}{(n-q+2)(n-q+4)\cdots(n-q+2c)}. \tag{8.8}$$

If P happens to be a submanifold of $\mathbb{R}^n$, then $V_P^{\mathbb{R}^n}(r)$ can be interpreted as the volume of a tube of radius r in $\mathbb{R}^n$. But if P is not given as a submanifold of $\mathbb{R}^n$, then (8.8) still makes sense and can be used to define a function $(P, n, r) \longmapsto V_P^{\mathbb{R}^n}(r)$. So, we can view $(P, n, r) \longmapsto V_P^{\mathbb{R}^n}(r)$ as an abstract function of curvature that has a geometric meaning as the tube volume whenever $P \subset \mathbb{R}^n$. This is the way that Corollary 8.6 should be interpreted.

It should be stressed that Theorem 8.4, Theorem 8.5 and Corollary 8.6 are global results, because the magnitude of r is limited only by the global geometry of P and M. On the other hand, we shall see in Corollary 9.25 that for *sufficiently small* r the dimension restrictions in Corollary 8.6 are unnecessary.

We shall prove Theorems 8.4 and 8.5 in several steps. First, we need to integrate some Riccati inequalities on the real line.

Lemma 8.7. *Assume f is differentiable on $(0, t_1)$ and continuous on $[0, t_1)$.*

(i) *Suppose $f' \geq f^2$ on $(0, t_1)$; then for $0 \leq t < t_1$ the following inequalities hold:*

$$f(t) \geq \frac{f(0)}{1 - f(0)t}, \tag{8.9}$$

$$1 - f(0)t > 0. \tag{8.10}$$

(ii) *Suppose that on $(0, t_1)$ both $f' \leq f^2$ and (8.10) hold. Then for $0 \leq t < t_1$ we have*

$$f(t) \leq \frac{f(0)}{1 - f(0)t}. \tag{8.11}$$

Proof. Define

$$g(t) = \Big(\big(1 - f(0)t\big) f(t) - f(0) \Big) \exp\Big(- \int_0^t f(s)\, ds \Big).$$

Then

$$g'(t) = \Big(1 - f(0)t\Big)\Big(f'(t) - f(t)^2\Big) \exp\Big(- \int_0^t f(s)\, ds \Big).$$

In case (ii) we have $g'(t) \leq 0$, so that g is nonincreasing. Since $g(0) = 0$, we conclude that $g(t) \leq 0$. This is just (8.11).

The same argument works with the inequalities reversed for case (i), provided we continue to assume (8.10). Suppose t_0 is a number such that $0 < t_0 \leq t_1$ and (8.10) holds for $0 \leq t < t_0$, but $f(0)t_0 = 1$. Then (8.9) is valid on $[0, t_0)$. As $t \longrightarrow t_0$ the denominator of the right-hand side of (8.9) goes to zero, so that $f(t) \longrightarrow +\infty$. Because f was assumed to be continuous on $[0, t_1)$, it follows that $t_0 = t_1$. Thus both (8.9) and (8.10) are consequences of the differential inequality $f' \geq f^2$. □

Lemma 8.7 will be applied to the principal curvature functions with finite initial values. To take care of the other principal curvature functions we need:

Lemma 8.8. *Suppose f is differentiable on $(0, t_1)$ and that $f(t) \longrightarrow -\infty$ as $t \longrightarrow 0$.*

(i) *If $f' \geq f^2$ on $(0, t_1)$, then $f(t) \geq -\dfrac{1}{t}$.*

(ii) *If $f' \leq f^2$ on $(0, t_1)$, then $f(t) \leq -\dfrac{1}{t}$.*

Proof. Let $0 < \varepsilon < t_1$ and put $f_\varepsilon(t) = f(t+\varepsilon)$. Then f_ε is defined on $[0, t_1 - \varepsilon)$. In case (i) we have $f_\varepsilon' \geq f_\varepsilon^2$, so we can use Lemma 8.7 on the function f_ε to conclude that

$$f_\varepsilon(t) \geq \frac{f_\varepsilon(0)}{1 - f_\varepsilon(0)t} = \frac{1}{\dfrac{1}{f_\varepsilon(0)} - t}$$

for $0 \leq t < t_1 - \varepsilon$. Letting $\varepsilon \longrightarrow 0$ we obtain $f(t) \geq -\dfrac{1}{t}$.

To prove (ii), we first notice that (8.10) holds for the function f_ε for sufficiently small $\varepsilon > 0$. The rest of the proof of (ii) is the same as that of (i) except all of the inequalities are reversed. □

For $m \in \exp_\nu(\mathcal{O}_P) - P$ let ξ be a unit-speed geodesic from m meeting P orthogonally at $\xi(0)$. Put $u = \xi'(0) \in P^\perp_{\xi(0)}$, and let t be the real number such that $m = \xi(t)$. Also, we define

$$\kappa(u) = \sup\{\, \langle T_u x, x\rangle \mid \|x\| = 1,\ x \in P_p \,\} = \text{the maximum eigenvalue of } T_u.$$

We now obtain some inequalities for $\operatorname{tr} S(t)$ that depend on the sectional curvature K^M of M. These inequalities generalize the inequality (3.35), which is the estimate we obtained for $\operatorname{tr} S(t)$ in the case that P is a point.

Lemma 8.9. *Assume that $K^M \geq 0$. Then*

$$\kappa(u) e_c(p, u) \leq 1, \tag{8.12}$$

and on $(0, e_c(p, u))$ we have

$$\operatorname{tr} S(t) \geq \operatorname{tr}\left(\frac{T_u}{I - tT_u}\right) - \frac{n - q - 1}{t}. \tag{8.13}$$

Proof. Let E be a parallel vector field along ξ of unit length and write $f(t) = \langle SE, E\rangle(t)$. It will be convenient to use the notation of Corollary 3.5: $S(t)$ and $R(t)$ are the restrictions of S and R_N to $\xi(t)$ where $N_{\xi(t)} = \xi'(t)$. Then $\langle RE, E\rangle$ is just the sectional curvature of the plane spanned by N and E. By Corollary 3.5, the assumption that $K^M \geq 0$ and the Cauchy-Schwarz Inequality, we have

$$\begin{aligned} f' &= \langle SE, E\rangle' = \langle S'E, E\rangle = \big\langle (S^2 + R)E, E\big\rangle \\ &\geq \big\langle S^2E, E\big\rangle = \|SE\|^2 \geq \langle SE, E\rangle^2 = f^2. \end{aligned}$$

First, we choose E so that $E(0) \in P_{\xi(0)}$ and write $e = E(0)$; then $f(0) = T_{eeu}$. By Lemma 8.7 we get

$$\langle SE, E\rangle(t) \geq \frac{T_{eeu}}{1 - tT_{eeu}} \tag{8.14}$$

and

$$1 - tT_{eeu} > 0 \tag{8.15}$$

for $0 \leq t \leq e_c(p,u)$. Since (8.15) holds for all unit vectors $e \in P_{\xi(0)}$, the inequality (8.12) follows.

Similarly, for the choice $e = E(0) \in P^{\perp}_{\xi(0)}$ with $\langle e, \xi'(0)\rangle = 0$ we have $f(0) = -\infty$, and so by Lemma 8.8

$$\langle SE, E\rangle(t) \geq -\frac{1}{t} \tag{8.16}$$

for $0 \leq t \leq e_c(p,u)$.

Now we choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of $M_{\xi(0)}$ such that the vectors $e_1, \ldots, e_q$ form a basis of $P_{\xi(0)}$ and $e_{q+1} = u$. Suppose that $t \longmapsto \{E_1(t), \ldots, E_n(t)\}$ is a parallel orthonormal frame field along ξ such that $E_\alpha(0) = e_\alpha$ for $1 \leq \alpha \leq n$. Then we have

$$\operatorname{tr} S(t) = \sum_{\substack{\alpha=1 \\ \alpha \neq q+1}}^{n} \langle SE_\alpha, E_\alpha\rangle(t). \tag{8.17}$$

Now (8.13) follows from (8.14), (8.16) and (8.17). $\square$

This proof does not work when $K^M \leq 0$ with reversed inequalities, because at a crucial point in Lemma 8.9 the Cauchy-Schwarz Inequality was invoked. Still, with stronger hypotheses it is possible to obtain an inequality:

Lemma 8.10. *Assume that $K^M \leq 0$. Suppose that $S(t)$ and its eigenvectors are defined and differentiable for $0 \leq t \leq e_c(p,u)$, and that (8.12) holds. Then on $(0, e_c(p,u))$, we have*

$$\operatorname{tr} S(t) \leq \operatorname{tr}\left(\frac{T_u}{I - tT_u}\right) - \frac{n-q-1}{t}. \tag{8.18}$$

Proof. Let $t \longmapsto \{F_1(t), \dots, F_n(t)\}$ be a frame field along ξ such that

$$SF_\alpha = \kappa_\alpha F_\alpha \tag{8.19}$$

for $\alpha \neq q+1$ and $F_1(0), \dots, F_q(0)$ are tangent to P. By Corollary 3.5 and the assumption that $K^M \leq 0$, we see that each of the principal curvature functions κ_α satisfies a Riccati inequality, namely $\kappa_\alpha' \leq \kappa_\alpha^2$. Condition (8.12) guarantees that $1 - t\kappa_a(0) > 0$ for $a = 1, \dots, q$, provided $0 \leq t \leq e_c(p, u)$. Furthermore, using Lemmas 8.7 and 8.8, we can solve the inequality $\kappa_\alpha' \leq \kappa_\alpha^2$ to get

$$\begin{cases} \kappa_a(t) \leq \dfrac{\kappa_a(0)}{1 - t\kappa_a(0)} & \text{for } a = 1, \dots, q, \\[2ex] \kappa_i(t) \leq -\dfrac{1}{t} & \text{for } i = q+2, \dots, n. \end{cases} \tag{8.20}$$

Since

$$\operatorname{tr} S(t) = \sum_{a=1}^{q} \kappa_a(t) + \sum_{i=q+2}^{n} \kappa_i(t),$$

we obtain (8.18) from (8.19) and (8.20). $\square$

We must interrupt our study of Riccati inequalities for a lemma from linear algebra.

Lemma 8.11. *Let A be any matrix. Then* $\det(e^A) = e^{\operatorname{tr}(A)}$.

Proof. First, let B be a diagonalizable matrix, and let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of B. Then $e^{\lambda_1}, \dots, e^{\lambda_n}$ are the eigenvalues of e^B, and so

$$\det(e^B) = e^{\lambda_1} \cdots e^{\lambda_n} = e^{\lambda_1 + \cdots + \lambda_n} = e^{\operatorname{tr}(B)}.$$

Furthermore, if N is a nilpotent matrix, then N is similar to an (upper triangular) matrix with zeros on the diagonal, and so

$$\det(e^N) = 1 = e^{\operatorname{tr}(N)}.$$

It follows from the Jordan Decomposition Theorem that for a general matrix A, we can write $A = B + N$ where B and N are commuting matrices with B diagonalizable and N nilpotent. Then

$$\det(e^A) = \det(e^B)\det(e^N) = e^{\operatorname{tr}(B)} = e^{\operatorname{tr}(A)}. \qquad \square$$

We are now ready to give an estimate Θ_u in terms of the Weingarten map T_u.

Lemma 8.12. *Assume that* $K^M \geq 0$.

(i) *For all* $t \geq 0$

$$t \longmapsto \frac{\Theta_u(t)}{\max(\det(I - t\,T_u), 0)}$$

is a nonincreasing function, and

$$\Theta_u(t) \leq \max(\det(I - t\,T_u), 0) \leq \max\left(\left(1 - \frac{t}{q}\langle H, u\rangle\right)^q, 0\right). \tag{8.21}$$

(ii) *In the range* $0 \leq t \leq e_c(p,u)$ *we have*

$$\Theta_u(t) \leq \det(I - t\,T_u) \leq \left(1 - \frac{t}{q}\langle H, u\rangle\right)^q, \tag{8.22}$$

so that the first zero of $\det(I - tT_u)$ *does not occur before the first zero of* $\Theta_u(t)$.

Proof. First assume that $0 \leq t < e_c(p,u)$. Since (8.12) holds, we have

$$t\,\kappa_\alpha \leq t\,\kappa(u) \leq e_c(p,u)\kappa(u) < 1$$

for any eigenvalue κ_α of T_u; consequently the map $I - tT_u$ has positive determinant. From Theorem 3.11 and Lemma 8.9 we obtain

$$\begin{aligned} \frac{d}{dt}\log\Theta_u(t) &= \frac{d}{dt}\log\vartheta_u(t) = \frac{\vartheta_u'(t)}{\vartheta_u(t)} \qquad (8.23) \\ &= -\left(\frac{n-q-1}{t} + \operatorname{tr} S(t)\right) \\ &\leq -\operatorname{tr}\left(\frac{T_u}{I - t\,T_u}\right) = \frac{d}{dt}\operatorname{tr}\big(\log(I - t\,T_u)\big). \end{aligned}$$

We know that we can write $I - tT_u = e^A$ for some linear transformation A; then Lemma 8.11 implies that

$$\operatorname{tr}\big(\log(I - t\,T_u)\big) = \operatorname{tr}(A) = \log\det(e^A) = \log\det(I - t\,T_u). \tag{8.24}$$

It follows from (8.23) and (8.24) that

$$\frac{d}{dt}\log\left(\frac{\Theta_u(t)}{\det(I - t\,T_u)}\right) \leq 0.$$

Since $\Theta_u(0) = 1$, we see that $t \longmapsto \Theta_u(t)\det(I - t\,T_u)^{-1}$ is nonincreasing. Therefore, we get the first inequality of (8.21), provided that we make the assumption that $0 \leq t < e_c(p,u)$. In fact, in the range $0 \leq t \leq e_c(p,u)$ we have the first inequality of (8.22), namely

$$\Theta_u(t) \leq \det(I - t\,T_u).$$

From this inequality we see that the first zero of $\det(I - tT_u)$ cannot occur before the first zero of $\Theta_u(t)$.

Because $\Theta_u(t)$ vanishes for $t \geq e_c(p,u)$, it follows that

$$t \longmapsto \frac{\Theta_u(t)}{\max(\det(I - t\,T_u), 0)}$$

is nonincreasing for the entire range $0 \leq t < \infty$. In particular,

$$\frac{\Theta_u(t)}{\max(\det(I - t\,T_u), 0)} \leq 1. \tag{8.25}$$

Thus the first inequality of (8.21) holds for all $t \geq 0$.

The second inequality of (8.21), as well as the second inequality of (8.22), follows from the inequality between the geometric and arithmetic means:

$$\begin{aligned}\Big(\det(I - t\,T_u)\Big)^{1/q} &= \left(\prod_{a=1}^{q}(1 - t\kappa_a(0))\right)^{1/q}\\ &\leq \frac{1}{q}\sum_{a=1}^{q}\Big(1 - t\kappa_a(0)\Big) = 1 - \frac{t}{q}\langle H, u\rangle.\end{aligned}$$ □

When we integrate (8.22) over the unit sphere in $P_p^{\perp}$, we obtain an estimate for the average of Θ_u entirely in terms of curvature.

Lemma 8.13. *Assume* $K^M \geq 0$.

(i) *For* $0 \leq t \leq e_c(p,u)$ *for every* $u \in P_p^{\perp}, u = 1$, *we have*

$$\int_0^r \int_{S^{n-q-1}(1)} t^{n-q-1}\Theta_u(t)\,du\,dt \tag{8.26}$$

$$\leq \frac{(\pi r^2)^{(n-q)/2}}{(\frac{1}{2}(n-q))!}\sum_{c=0}^{[q/2]}\frac{C^{2c}((R^P - R^M)^c)r^{2c}}{c!(2c)!(n-q+2)(n-q+4)\cdots(n-q+2c)}$$

$$\leq \frac{(\pi r^2)^{(n-q)/2}}{(\frac{1}{2}(n-q))!}\sum_{c=0}^{[q/2]}\frac{q!\|H\|^{2c}r^{2c}}{2^c q^{2c} c!(q-2c)!(n-q+2)(n-q+4)\cdots(n-q+2c)}.$$

(ii) *For all* $t \geq 0$ *we have*

$$\begin{aligned}\int_{S^{n-q-1}(1)}\Theta_u(t)\,du &\leq \int_{S^{n-q-1}(1)}\max(\det(I - t\,T_u), 0)\,du \qquad (8.27)\\ &\leq \int_{S^{n-q-1}(1)}\max\left(\left(1 - \frac{t}{q}\langle H, u\rangle\right)^q, 0\right)du.\end{aligned}$$

Proof. Integration of (8.21) over $S^{n-q-1}(1)$ yields (8.27). Next suppose that $0 \leq t \leq e_c(p,u)$. From Theorem 4.10 we know that

$$\int_0^r \int_{S^{n-q-1}(1)} t^{n-q-1} \det(\delta_{ab} - t\,T_{abu})\,du\,dt \tag{8.28}$$

$$= \frac{(\pi r^2)^{(n-q)/2}}{(\frac{1}{2}(n-q))!} \sum_{c=0}^{[q/2]} \frac{C^{2c}((R^P - R^M)^c) r^{2c}}{c!(2c)!(n-q+2)(n-q+4)\cdots(n-q+2c)}.$$

Then the first inequality of (8.26) follows from (8.22) and (8.28).

To prove the second inequality of (8.26), we use Corollary 4.6 to integrate $\langle H, u\rangle^{2c}$ over $S^{n-q-1}(1)$:

$$\int_{S^{n-q-1}(1)} \langle H, u\rangle^{2c}\,du \tag{8.29}$$

$$= \frac{2\pi^{(n-q)/2} 1\cdot 3\cdots(2c-1)}{\Gamma(\frac{1}{2}(n-q))(n-q)\cdots(n-q+2c-2)} \sum_{i_1\ldots i_c=q+1}^{n} \langle H, e_{i_1}\rangle^2 \cdots \langle H, e_{i_c}\rangle^2$$

$$= \frac{2\pi^{(n-q)/2}(2c)!\|H\|^{2c}}{4^c c!\,\Gamma(\frac{1}{2}(n-q+2c))}.$$

Making use of (8.29) and the binomial theorem, we compute

$$\int_0^r t^{n-q-1} \int_{S^{n-q-1}(1)} \left(1 - \frac{t}{q}\langle H, u\rangle\right)^q du\,dt \tag{8.30}$$

$$\leq \sum_{c=0}^{[q/2]} \binom{q}{2c} \left(\frac{1}{q}\right)^{2c} \int_0^r t^{n-q-1+2c} dt \int_{S^{n-q-1}(1)} \langle H, u\rangle^{2c} du$$

$$= \frac{(\pi r^2)^{(n-q)/2}}{(\frac{1}{2}(n-q))!} \sum_{c=0}^{[q/2]} \frac{q!\|H\|^{2c} r^{2c}}{2^c q^{2c}(q-2c)!\,c!\,(n-q+2)\cdots(n-q+2c)}.$$

Then the second inequality of (8.26) follows from (8.28) and (8.30). □

Similarly, using Theorem 3.11 and Lemma 8.10, we get nonpositive curvature versions of Lemma 8.12 and Lemma 8.13.

Lemma 8.14. *Assume that* $K^M \leq 0$. *Suppose that* $S(t)$ *and its eigenvectors are defined and differentiable for* $0 \leq t \leq e_c(p,u)$, *and that* (8.12) *holds. Then on* $(0, e_c(p,u))$

$$t \longmapsto \frac{\Theta_u(t)}{\det(I - t\,T_u)}$$

is a nondecreasing function and $\Theta_u(t) \geq \det(I - t\,T_u)$.

Lemma 8.15. *Assume the hypotheses of Lemma* 8.14. *Then for* $0 \le r \le e_c(p,u)$ *we have*

$$\int_0^r \int_{S^{n-q-1}(1)} t^{n-q-1}\Theta_u(t)\,du\,dt \tag{8.31}$$

$$\ge \frac{(\pi r^2)^{(n-q)/2}}{(\frac{1}{2}(n-q))!}\sum_{c=0}^{[q/2]} \frac{C^{2c}((R^P - R^M)^c)r^{2c}}{c!(2c)!(n-q+2)(n-q+4)\cdots(n-q+2c)}.$$

Proof of Theorems 8.4 and 8.5. Suppose $K^M \ge 0$. To prove (8.5), we integrate (8.26) over P. It follows from Lemmas 8.2 and 8.3 that the integral of the left-hand side of (8.26) is $V_P^M(r)$. Thus we obtain (8.5). In the same way when we integrate (8.27) over P we get (8.6).

The proof of (8.7) for the case when $K^M \le 0$ and the principal curvature functions are distinct is the same as that of the first inequality of (8.5), but with all the inequalities reversed. If there is some problem with the differentiability of the principal curvature functions, the submanifold can be joggled slightly. Then (8.7) holds for the joggled submanifold. Since the volume function is a continuous function of the submanifold, (8.7) holds for the original submanifold. □

Proof of Corollary 8.6. Suppose $\dim P \le 3$ and $0 \le r \le \operatorname{minfoc}(P)$. Then the right-hand side of the first inequality of (8.5) has at most two terms. If $q = \dim P$ is 0 or 1 and $K^M \ge 0$, then

$$V_P^M(r) \le \frac{(\pi r^2)^{(n-q)/2}}{(\frac{1}{2}(n-q))!}\operatorname{volume}(P) = V_P^{\mathbb{R}^n}(r).$$

Similarly, if $q = \dim P$ is 2 or 3 and $K^M \ge 0$, then

$$V_P^M(r) \le \frac{(\pi r^2)^{(n-q)/2}}{(\frac{1}{2}(n-q))!}\left\{\operatorname{volume}(P) + \frac{r^2}{2(n-q+2)}\int_P \tau(R^P - R^M)\,dP\right\}.$$

But

$$\tau(R^P - R^M) = \tau(R^P) - \sum_{ab=1}^{q} R^M_{abab} \le \tau(R^P),$$

and we again get

$$V_P^M(r) \le V_P^{\mathbb{R}^n}(r).$$

The proof of part (ii) is analogous. □

This proof fails when $\dim P \ge 4$. The problem is that the curvature invariants beyond the scalar curvature are nonlinear functions of curvature. This dimension problem does not arise for small r, as we shall see in Corollary 9.25.

8.3 The Bishop-Günther Inequalities Generalized to Tubes

In this section we estimate the volume of a tube in a Riemannian manifold M whose sectional curvature satisfies $K^M \geq \lambda$ or $K^M \leq \lambda$. Here, λ will denote a constant, which can be positive, negative or zero. We state the theorems only in the case that $\lambda > 0$, but the corresponding statements for $\lambda = 0$ or $\lambda < 0$ can be deduced easily.

When the submanifold is a point, the Bishop-Günther Inequalities (Theorem 3.17) hold, and so it is reasonable to expect corresponding inequalities for tubes. The situation is technically more complicated than Theorem 8.4, however. Also, there are some integrals of $\sin^{n-1}(t\sqrt{\lambda})$ whose computations are inconvenient to carry out (but see Section A.3 of the Appendix). On the other hand, we can write down explicit inequalities for $A_P^M(r)$. The corresponding inequalities for $V_P^M(r)$ can be obtained by integration via Lemmas 3.16 and 8.3.

The following two theorems, whose proof are given later in this section, generalize Theorems 8.4 and 8.5:

Theorem 8.16. *Suppose $K^M \geq \lambda$.*

(i) *For $0 \leq r \leq \mathrm{minfoc}(P)$*

$$A_P^M(r) \leq \frac{2\pi^{(n-q)/2}}{\Gamma(\frac{1}{2}(n-q))} \sum_{c=0}^{[q/2]} \frac{k_{2c}(R^P - R^M)}{(n-q)(n-q+2)\cdots(n-q+2c-2)} \tag{8.32}$$
$$\cdot \left(\cos(r\sqrt{\lambda})\right)^{q-2c} \left(\frac{\sin(r\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-q+2c-1}$$

$$\leq \int_P \int_{S^{n-q-1}(1)} \left(\frac{\sin(r\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-q-1}$$
$$\cdot \left(\cos(r\sqrt{\lambda}) - \frac{\sin(r\sqrt{\lambda})}{q\sqrt{\lambda}}\langle H, u\rangle\right)^q du\, dP$$

$$= \sum_{c=0}^{[q/2]} \frac{2\pi^{(n-q)/2} q!}{(q-2c)!\, c!\, \Gamma(\frac{1}{2}(n-q+2c))} (\cos(r\sqrt{\lambda}))^{q-2c}$$
$$\cdot \left(\frac{\sin(r\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-q-1+2c} \int_P \left(\frac{1}{q}\|H\|\right)^{2c} dP.$$

(ii) *For all* $r \geq 0$

$$A_P^M(r) \leq \int_P \int_{S^{n-q-1}(1)} \max\left(\left(\frac{\sin(r\sqrt{\lambda})}{r\sqrt{\lambda}} \right)^{n-q-1} \cdot \det\left(\cos(r\sqrt{\lambda}) I - \frac{\sin(r\sqrt{\lambda})}{\sqrt{\lambda}} T_u \right), 0 \right) du\, dP \tag{8.33}$$

$$\leq \int_P \int_{S^{n-q-1}(1)} \max\left(\left(\frac{\sin(r\sqrt{\lambda})}{r\sqrt{\lambda}} \right)^{n-q-1} \cdot \left(\cos(r\sqrt{\lambda}) - \frac{\sin(r\sqrt{\lambda})}{q\sqrt{\lambda}} \langle H, u \rangle \right)^q, 0 \right) du\, dP.$$

The result for $K^M \leq \lambda$ corresponding to Theorem 8.16 is weaker:

Theorem 8.17. *If* $K^M \leq \lambda$, *then for* $0 \leq r \leq \mathrm{minfoc}(P)$

$$A_P^M(r) \geq \frac{2\pi^{(n-q)/2}}{\Gamma(\frac{1}{2}(n-q))} \sum_{c=0}^{[q/2]} \frac{k_{2c}(R^P - R^M)}{(n-q)(n-q+2)\cdots(n-q+2c-2)} \cdot \left(\cos(r\sqrt{\lambda})\right)^{q-2c} \left(\frac{\sin(r\sqrt{\lambda})}{\sqrt{\lambda}} \right)^{n-q+2c-1}. \tag{8.34}$$

The proofs of Theorems 8.16 and 8.17 are quite similar to those of Theorems 8.4 and 8.5, just more complicated. First, we need lemmas that generalize Lemmas 8.7 and 8.8:

Lemma 8.18. *Assume* f *is differentiable on* $(0, t_1)$ *and continuous on* $[0, t_1)$.

(i) *Suppose* $f' \geq f^2 + \lambda$ *on* $(0, t_1)$; *then for* $0 \leq t < t_1$ *the following inequalities hold:*

$$f(t) \geq \frac{\sqrt{\lambda}\sin(t\sqrt{\lambda}) + f(0)\cos(t\sqrt{\lambda})}{\cos(t\sqrt{\lambda}) - \dfrac{f(0)}{\sqrt{\lambda}}\sin(t\sqrt{\lambda})}, \tag{8.35}$$

$$\cos(t\sqrt{\lambda}) - \frac{f(0)}{\sqrt{\lambda}}\sin(t\sqrt{\lambda}) > 0. \tag{8.36}$$

(ii) *Suppose that on* $(0, t_1)$ *both* $f' \leq f^2 + \lambda$ *and* (8.36) *hold. Then for* $0 \leq t < t_1$ *we have*

$$f(t) \leq \frac{\sqrt{\lambda}\sin(t\sqrt{\lambda}) + f(0)\cos(t\sqrt{\lambda})}{\cos(t\sqrt{\lambda}) - \dfrac{f(0)}{\sqrt{\lambda}}\sin(t\sqrt{\lambda})}.$$

Proof. Define

$$p(t) = \left\{\left(\cos(t\sqrt{\lambda}) - \frac{f(0)}{\sqrt{\lambda}}\sin(t\sqrt{\lambda})\right)f(t) - f(0)\cos(t\sqrt{\lambda}) - \sqrt{\lambda}\sin(t\sqrt{\lambda})\right\} \cdot \exp\left(-\int_0^t f(s)\,ds\right).$$

The proof of Lemma 8.18 is the same as that of Lemma 8.7; the only difference is that $g(t)$ is replaced by $p(t)$. □

Remark. The right-hand side of (8.35) can be written as

$$\sqrt{\lambda}\tan\left(\arctan\left(\frac{f(0)}{\sqrt{\lambda}}\right) + t\sqrt{\lambda}\right).$$

Lemma 8.19. *Suppose f is differentiable on $(0, t_1)$ and $f(t) \longrightarrow -\infty$ as $t \longrightarrow 0$.*

(i) *If $f' \geq f^2 + \lambda$, then on $(0, t_1)$ we have*

$$f(t) \geq \frac{-\sqrt{\lambda}}{\tan(t\sqrt{\lambda})}. \tag{8.37}$$

(ii) *Suppose $f' \leq f^2 + \lambda$. In the case that $\lambda > 0$ assume that*

$$t_1 < \frac{\pi}{2\sqrt{\lambda}} \tag{8.38}$$

holds. Then on $(0, t_1)$ we have

$$f(t) \leq \frac{-\sqrt{\lambda}}{\tan(t\sqrt{\lambda})}. \tag{8.39}$$

Proof. Lemma 8.19 follows from Lemma 8.18 in exactly the same way that Lemma 8.8 follows from Lemma 8.7; we give the details. Let $0 < \varepsilon < t_1$ and put $f_\varepsilon(t) = f(t+\varepsilon)$. Then $f_\varepsilon' \geq f_\varepsilon^2 + \lambda$, so that we can use Lemma 8.18 on the function f_ε. Thus in case (i) we have

$$f(t+\varepsilon) = f_\varepsilon(t) \geq \frac{f_\varepsilon(0) + \sqrt{\lambda}\tan(t\sqrt{\lambda})}{1 - \dfrac{f_\varepsilon(0)}{\sqrt{\lambda}}\tan(t\sqrt{\lambda})} = \frac{1 + \dfrac{\sqrt{\lambda}\tan(t\sqrt{\lambda})}{f(\varepsilon)}}{\dfrac{1}{f(\varepsilon)} - \dfrac{\tan(t\sqrt{\lambda})}{\sqrt{\lambda}}}$$

for $0 \leq t \leq t_1 - \varepsilon$. Letting $\varepsilon \longrightarrow 0$ we obtain (8.37). This proves (i).

We do part (ii) in the case that $\lambda > 0$. Assumption (8.38) implies that both $\cos(t\sqrt{\lambda})$ and $\sin(t\sqrt{\lambda})$ are positive. Therefore, we can choose $\varepsilon > 0$ small enough so that

$$\cos(t\sqrt{\lambda}) - \frac{f_\varepsilon(0)}{\sqrt{\lambda}}\sin(t\sqrt{\lambda}) > 0.$$

From part (i) of Lemma 8.18 we get (8.39) by reversing the inequalities in the proof of (8.37). □

We are now ready to generalize the estimates for $e_c(p,u)$ and $S(t)$ that we obtained in Lemma 8.18.

Lemma 8.20. *Assume that $K^M \geq \lambda$. Then*

$$\frac{\kappa(u)\tan(e_c(p,u)\sqrt{\lambda})}{\sqrt{\lambda}} \leq 1, \tag{8.40}$$

and on $(0, e_c(p,u))$ we have

$$\operatorname{tr} S(t) \geq \operatorname{tr}\left(\frac{\sqrt{\lambda}\tan(t\sqrt{\lambda})\, I + T_u}{I - \dfrac{1}{\sqrt{\lambda}}\tan(t\sqrt{\lambda})\, T_u}\right) - \frac{(n-q-1)\sqrt{\lambda}}{\tan(t\sqrt{\lambda})}.$$

Proof. The proof of Lemma 8.20 is essentially that of Lemma 8.9. The only difference is that the first parts of Lemmas 8.18 and 8.19 are used in place of the first parts of Lemmas 8.7 and 8.9. □

There is a similar generalization of Lemma 8.10. It is proved by using the second parts of Lemmas 8.18 and 8.19 in place of the second parts of Lemmas 8.7 and 8.8.

Lemma 8.21. *Assume that $K^M \leq \lambda$, and that $S(t)$ and its eigenvectors are defined and differentiable for $0 \leq t \leq e_c(p,u)$, and that (8.40) holds. If $\lambda > 0$ assume that $e_c(p,u) \leq \pi/2\sqrt{\lambda}$. Then on $(0, e_c(p,u))$ we have*

$$\operatorname{tr} S(t) \leq \operatorname{tr}\left(\frac{\sqrt{\lambda}\tan(t\sqrt{\lambda})\, I + T_u}{I - \dfrac{1}{\sqrt{\lambda}}\tan(t\sqrt{\lambda})\, T_u}\right) - \frac{(n-q-1)\sqrt{\lambda}}{\tan(t\sqrt{\lambda})}.$$

The proofs of the following two lemmas are omitted because they are straightforward generalizations of Lemmas 8.12 and 8.13.

Lemma 8.22. *Suppose that $K^M \geq \lambda$. Then for $0 \leq t \leq e_c(p,u)$*

$$t \longmapsto \Theta_u(t)\left\{\max\left(\left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{n-q-1}\det\left(\cos(t\sqrt{\lambda})I - \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}T_u\right), 0\right)\right\}^{-1}$$

is a nonincreasing function, and

$$\begin{aligned}\Theta_u(t) &\leq \max\left(\left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{n-q-1}\det\left(\cos(t\sqrt{\lambda})I - \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}T_u\right), 0\right)\\ &\leq \max\left(\left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{n-q-1}\left(\cos(t\sqrt{\lambda}) - \frac{\sin(t\sqrt{\lambda})}{q\sqrt{\lambda}}\langle H, u\rangle\right)^q, 0\right).\end{aligned} \tag{8.41}$$

Hence the first zero of $\Theta_u(t)$ *does not occur after that of*

$$\det\left(\cos(t\sqrt{\lambda})I - \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}T_u\right).$$

Lemma 8.23. *Assume* $K^M \geq \lambda$.

(i) *For* $0 \leq t \leq e_c(p, u)$ *we have*

$$\begin{aligned}&\int_{S^{n-q-1}(1)} t^{n-q-1}\Theta_u(t)\,du \\ &\leq \frac{2\pi^{(n-q)/2}}{\Gamma(\frac{1}{2}(n-q))}\sum_{c=0}^{[q/2]}\frac{C^c((R^P - R^M)^c)}{c!(2c)!(n-q)(n-q+2)\cdots(n-q+2c-2)}\\ &\qquad\cdot(\cos(t\sqrt{\lambda}))^{q-2c}\left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-q+2c-1}\\ &\leq \left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-q-1}\int_{S^{n-q-1}(1)}\left(\cos(t\sqrt{\lambda}) - \frac{\sin(t\sqrt{\lambda})}{q\sqrt{\lambda}}\langle H, u\rangle\right)^q du\\ &= \frac{2\pi^{(n-q)/2}}{\Gamma(\frac{1}{2}(n-q))}\sum_{c=0}^{[q/2]}\frac{q!\|H\|^{2c}}{2^c c^{2c}(q-2c)!\,c!\,(n-q)\cdots(n-q+2c-2)}\\ &\qquad\cdot(\cos(t\sqrt{\lambda}))^{q-2c}\left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-q-1+2c}.\end{aligned} \tag{8.42}$$

(ii) *For all* $t \geq 0$ *we have*

$$\begin{aligned}&\int_{S^{n-q-1}(1)} \Theta_u(t)\,du \\ &\leq \int_{S^{n-q-1}(1)} \max\left(\left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-q-1}\det\left(\cos(t\sqrt{\lambda})I - \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}T_u\right), 0\right)du\end{aligned} \tag{8.43}$$

$$\leq \int_{S^{n-q-1}(1)} \max\left(\left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-q-1}\left(\cos(t\sqrt{\lambda}) - \frac{\sin(t\sqrt{\lambda})}{q\sqrt{\lambda}}\langle H, u\rangle\right)^q, 0\right) du.$$

Similarly, there are generalizations of Lemmas 8.14 and 8.15:

Lemma 8.24. *Assume that $K^M \leq \lambda$. Suppose that $S(t)$ and its eigenvectors are defined and differentiable for $0 \leq t \leq e_c(p,u)$, and that (8.40) holds. If $\lambda > 0$ assume that $e_c(p,u) \leq \pi/2\sqrt{\lambda}$. Then for $0 \leq t < e_c(p,u)$*

$$t \longmapsto \Theta_u(t)\left\{\left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{n-q-1} \det\left(\cos(t\sqrt{\lambda})I - \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}T_u\right)\right\}^{-1}$$

is a nondecreasing function, and

$$\Theta_u(t) \geq \left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{n-q-1} \det\left(\cos(t\sqrt{\lambda})I - \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}T_u\right).$$

Lemma 8.25. *Assume $K^M \leq \lambda$. For $0 \leq r \leq e_c(p,u)$*

$$\int_{S^{n-q-1}(1)} r^{n-q-1}\Theta_u(r)\, du \tag{8.44}$$

$$\geq \frac{2\pi^{(n-q)/2}}{\Gamma(\frac{1}{2}(n-q))} \sum_{c=0}^{[q/2]} \frac{C^c((R^P - R^M)^c)}{c!(2c)!(n-q)(n-q+2)\cdots(n-q+2c-2)}$$

$$\cdot \left(\cos(r\sqrt{\lambda})\right)^{q-2c} \left(\frac{\sin(r\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-q+2c-1}.$$

As a special case we obtain the formula for the infinitesimal change of volume function for a submanifold of a space of constant sectional curvature.

Corollary 8.26. *Suppose that P is a topologically embedded submanifold with compact closure in a space $\mathbb{K}^n(\lambda)$ of constant sectional curvature λ. Then for* $0 \leq r \leq \operatorname{minfoc}(P)$, *the infinitesimal change of volume function is given by*

$$\vartheta_u(t) = \Theta_u(t) = \left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{n-q-1} \det\left(\cos(t\sqrt{\lambda})I - \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}T_u\right)$$

$$\leq \left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-q-1}\left(\cos(t\sqrt{\lambda}) - \frac{\sin(t\sqrt{\lambda})}{q\sqrt{\lambda}}\langle H, u\rangle\right)^q.$$

Furthermore,

$$\int_{S^{n-q-1}(1)} r^{n-q-1}\Theta_u(r)\,du \tag{8.45}$$

$$= \frac{2\pi^{(n-q)/2}}{\Gamma(\frac{1}{2}(n-q))}\sum_{c=0}^{[q/2]} \frac{C^c((R^P - R^{\mathbb{K}^n(\lambda)})^c)}{c!(2c)!(n-q)(n-q+2)\cdots(n-q+2c-2)}$$

$$\cdot\left(\cos(r\sqrt{\lambda})\right)^{q-2c}\left(\frac{\sin(r\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-q+2c-1}$$

$$\leq \left(\frac{\sin(r\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-q-1}\int_{S^{n-q-1}(1)}\left(\cos(r\sqrt{\lambda}) - \frac{\sin(r\sqrt{\lambda})}{q\sqrt{\lambda}}\langle H, u\rangle\right)^q du$$

$$= \frac{2\pi^{(n-q)/2}}{\Gamma(\frac{1}{2}(n-q))}\sum_{c=0}^{[q/2]} \frac{q!\|H\|^{2c}}{2^c c^{2c}(q-2c)!\,c!\,(n-q)\cdots(n-q+2c-2)}$$

$$\cdot(\cos(r\sqrt{\lambda}))^{q-2c}\left(\frac{\sin(r\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-q-1+2c}.$$

Proof of Theorems 8.16 and 8.17. Suppose $K^M \geq \lambda$. We integrate (8.33) over P to obtain (8.32). The rest of the proofs of Theorems 8.16 and 8.17 is a straightforward generalization of the proofs of Theorems 8.4 and 8.5. □

Weyl's formula for the volume of a tube about a submanifold of a space $\mathbb{K}^n(\lambda)$ of constant curvature λ is a special case of the next theorem.[1]

Theorem 8.27. *Suppose that P is a topologically embedded submanifold with compact closure in a space $\mathbb{K}^n(\lambda)$ of constant sectional curvature λ. Then for $r \leq$ $\operatorname{minfoc}(P)$ we have*

$$A_P^{\mathbb{K}^n(\lambda)}(r) = \frac{2\pi^{(n-q)/2}}{\Gamma(\frac{1}{2}(n-q))}\sum_{c=0}^{[q/2]} \frac{k_{2c}(R^P - R^{\mathbb{K}^n(\lambda)})}{(n-q)(n-q+2)\cdots(n-q+2c-2)} \tag{8.46}$$

$$\cdot\left(\cos(r\sqrt{\lambda})\right)^{q-2c}\left(\frac{\sin(r\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-q+2c-1}.$$

[1] Actually, Weyl did only the case of submanifolds of a sphere. His technique is completely different: he used the natural embedding of the sphere in a Euclidean space and then used Euclidean techniques to get the spherical tube formula.

8.4 Tube Volume Estimates Involving Ricci Curvature

In this section we sharpen Bishop's Theorem (Theorem 3.19). Saying that the Ricci curvature satisfies

$$\rho^M(x,x) \geq (n-1)\lambda\|x\|^2 \tag{8.47}$$

for all tangent vectors x to M is weaker than saying that the sectional curvature satisfies $K^M \geq \lambda$. Consequently, we can expect that the tube volume estimates implied by (8.47) would not be as strong as those implied by the condition $K^M \geq \lambda$.

First, we show that (8.47) still yields estimates on the infinitesimal change of volume function, but much weaker than those of Lemma 8.22. Although there are results for submanifolds of all dimensions, the cases $\dim P = 0$ and $\dim P = n-1$ are the main interest. In the latter case the notion of mean curvature is useful. The mean curvature is a variant of the mean curvature vector field H defined in Section 6.7. If $u \in P_p^{\perp}$ we define the **mean curvature in the direction** u to be $\langle H(p), u\rangle$. When P is a hypersurface, there are two unit normal vectors $\pm u \in P_p^{\perp}$; in this case we fix u and put $h = \langle H(p), u\rangle$.

Lemma 8.28. *Suppose that the Ricci curvature of M satisfies* (8.47).

(i) *If $q = \dim P \leq n-2$, then*

$$\operatorname{tr} S(t) \geq \frac{-(n-1)\sqrt{\lambda}}{\tan(t\sqrt{\lambda})}, \tag{8.48}$$

and

$$t \longmapsto \Theta_u(t)\left\{\max\left(\left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{n-1} t^q, 0\right)\right\}^{-1} \tag{8.49}$$

is nonincreasing for all $p \in P$ and $u \in P_p^{\perp}$ with $\|u\| = 1$. all $t \geq 0$.

(ii) *If $\dim P = 0$, then in addition to* (i) *we have*

$$\Theta_u(t) \leq \left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{n-1} \tag{8.50}$$

for $0 \leq t \leq \pi/\sqrt{\lambda}$. Hence the first zero of $\Theta_u(t)$ is not greater than $\pi/\sqrt{\lambda}$.

(iii) *If $\dim P = n-1$, then*

$$\operatorname{tr} S(t) \geq \frac{h + (n-1)\sqrt{\lambda}\tan(t\sqrt{\lambda})}{1 - \dfrac{\tan(t\sqrt{\lambda})\, h}{(n-1)\sqrt{\lambda}}}. \tag{8.51}$$

Furthermore,

$$t \longmapsto \Theta_u(t)\left\{\max\left(\left(\cos(t\sqrt{\lambda})-\frac{h}{n-1}\left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)\right)^{n-1},0\right)\right\}^{-1} \tag{8.52}$$

is nonincreasing, and

$$\Theta_u(t) \leq \left(\cos(t\sqrt{\lambda})-\frac{h}{n-1}\left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)\right)^{n-1} \tag{8.53}$$

provided $\left(\cos(t\sqrt{\lambda})-\frac{h}{n-1}\left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)\right) \geq 0$. *Hence the first zero of* $\Theta_u(t)$ *is not greater than that of* $\left(\cos(t\sqrt{\lambda})-\frac{h}{n-1}\left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)\right)$.

Proof. For (i) we can assume without loss of generality that $0 < t \leq e_c(p,u)$. We let

$$f(t) = \frac{\operatorname{tr} S(t)}{n-1}, \tag{8.54}$$

and use the Cauchy-Schwarz Inequality as in the proof of Theorem 3.19; we get $f'(t) \geq f(t)^2 + \lambda$. Thus we obtain (8.48)). From (8.48) it follows that

$$\begin{aligned}
\frac{d}{dt}\log\Theta_u(t) = \frac{\vartheta_u'(t)}{\vartheta_u(t)} &= -\left(\operatorname{tr} S(t) + \frac{n-q-1}{t}\right) \\
&\leq \frac{(n-1)\sqrt{\lambda}\cos(t\sqrt{\lambda})}{\sin(t\sqrt{\lambda})} - \frac{n-q-1}{t} \\
&= \frac{d}{dt}\log\left\{\left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{n-1} t^q\right\},
\end{aligned}$$

and so (8.49) follows. When $q = 0$, equation (8.50) is immediate from (8.49).

For part (iii) we again define $f(t)$ by (8.54). Just as in part (i), we have $f'(t) \geq f(t)^2 + \lambda$, but now $f(0)$ is finite instead of infinite. In fact,

$$h = \operatorname{tr} S(0) = (n-1)f(0).$$

Therefore, from (8.35) we have

$$\frac{\operatorname{tr} S(t)}{n-1} = f(t) \geq \frac{\sqrt{\lambda}\sin(t\sqrt{\lambda}) + f(0)\cos(t\sqrt{\lambda})}{\cos(t\sqrt{\lambda}) - \dfrac{f(0)}{\sqrt{\lambda}}\sin(t\sqrt{\lambda})}$$

$$= \frac{\sqrt{\lambda}\sin(t\sqrt{\lambda}) + \dfrac{h\cos(t\sqrt{\lambda})}{n-1}}{\cos(t\sqrt{\lambda}) - \dfrac{h\,\sin(t\sqrt{\lambda})}{(n-1)\sqrt{\lambda}}}.$$

Thus we get (8.51). Then (8.51) implies that for $0 < t \leq e_c(p,u)$ we have

$$\begin{aligned}
\frac{d}{dt}\log\Theta_u(t) &= \frac{\vartheta_u'(t)}{\vartheta_u(t)} = -\operatorname{tr} S(t)\\
&\leq (n-1)\left(\frac{-\sqrt{\lambda}\sin(t\sqrt{\lambda}) - \dfrac{h\cos(t\sqrt{\lambda})}{n-1}}{\cos(t\sqrt{\lambda}) - \dfrac{h\sin(t\sqrt{\lambda})}{(n-1)\sqrt{\lambda}}}\right)\\
&= \frac{d}{dt}\log\left\{\left(\cos(t\sqrt{\lambda}) - \frac{h\sin(t\sqrt{\lambda})}{(n-1)\sqrt{\lambda}}\right)^{n-1}\right\}.
\end{aligned}$$

Hence the rest of (iii) follows. □

We are now ready to give the sharpened version of Bishop's Theorem (Theorem 3.19, page 47.).

Theorem 8.29. *Assume that the Ricci curvature of M satisfies* (8.47). *Let $\Theta_u(t)$ denote the infinitesimal change of volume function at a point $m \in M$. Then for all $t \geq 0$*

$$t \longmapsto \frac{\Theta_u(t)}{\max\left(\left(\dfrac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{n-1}, 0\right)} \tag{8.55}$$

is nonincreasing, and

$$\Theta_u(t) \leq \max\left(\left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{n-1}, 0\right). \tag{8.56}$$

Furthermore, for all $r \geq 0$ we have

$$V_m^M(r) \leq \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}\int_0^r \max\left(\left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-1}, 0\right) dt = V_m^{\mathbb{K}^n(\lambda)}(r). \tag{8.57}$$

Proof. We have already proved (8.55) and (8.56) in Lemma 8.28. When we integrate (8.56) over the unit sphere in M_m and then integrate from 0 to r, we get (8.57). □

There is also an estimate for the volume of a tube about a hypersurface.

Corollary 8.30. *Let P be a hypersurface of a Riemannian manifold M whose Ricci curvature satisfies* (8.47). *Then for all $r \geq 0$ we have*

$$\mathrm{A}_P^M(r) \leq \int_P \max\left(\left(\cos(r\sqrt{\lambda}) - \frac{h}{n-1}\frac{\sin(r\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-1}, 0\right) dP.$$

Proof. This follows from (8.3) and (8.53). □

8.5 Comparison Theorems for the Volumes of Tubes about Kähler Submanifolds

In this section we shall discuss Kähler analogs of Theorems 8.16 and 8.17. *Throughout the section M will denote a complete Kähler manifold of real dimension $2n$, and P will denote a topologically embedded complex submanifold of M with compact closure of real dimension $2q$.*

In Section 6.2 we defined holomorphic sectional curvature K_{hol} as the restriction of the sectional curvature of a Kähler manifold to 2-dimensional subspaces of tangent spaces spanned by vectors of the form x, Jx. Similarly, we define the **antiholomorphic sectional curvature** K_{ah} to be the restriction of the sectional curvature to subspaces of tangent spaces spanned by vectors of the form x, y, where x, Jx, y, Jy are mutually perpendicular.

First, we prove the Kähler analog of Lemma 8.20.

Lemma 8.31. *Suppose that the holomorphic and antiholomorphic sectional curvatures of M satisfy $K_{\mathrm{hol}}^M \geq 4\lambda$ and $K_{\mathrm{ah}}^M \geq \lambda$, where $\lambda \geq 0$. Then* (8.40) *is satisfied, and on $(0, e_c(p,u))$ we have*

$$\operatorname{tr} S(t) \;\geq\; \operatorname{tr}\left(\frac{\sqrt{\lambda}\tan(t\sqrt{\lambda})\,I + T_u}{I - \dfrac{1}{\sqrt{\lambda}}\tan(t\sqrt{\lambda})\,T_u}\right) - \frac{2(n-q-1)\sqrt{\lambda}}{\tan(t\sqrt{\lambda})} - \frac{2\sqrt{\lambda}}{\tan(2t\sqrt{\lambda})}. \tag{8.58}$$

Proof. We begin by following the proof of Lemma 7.8, but replacing equalities by inequalities as in the proof of Lemma 8.10.

Let N denote the unit normal to the tubular hypersurface

$$\{\, m' \in M \mid \mathrm{distance}(P, m') = t \,\}.$$

Let $u \in P_p^\perp$ with $\|u\| = 1$, and let ξ be a unit-speed geodesic in M with $\xi(0) = p$ and $\xi'(0) = u$. Because P is a complex submanifold of M there exists a holomorphic orthonormal frame $\{e_1, \ldots, Je_n\}$ with $e_{q+1} = u$ that diagonalizes the Weingarten map T_u on P_p. We choose a parallel orthonormal frame field $t \longmapsto \{E_1(t), \ldots, E_{n^*}(t)\}$ along ξ which coincides with $\{e_1, \ldots, Je_n\}$ at p. We define

$$\kappa_i(t) = \langle S(t)E_i(t), E_i(t)\rangle$$

for $i = 1, \ldots, n^*$, $i \neq q+1$. We obtain the following differential inequalities:

$$\kappa_a' \geq \kappa_a^2 + \lambda \qquad (a = 1, \ldots, q^*), \tag{8.59}$$

$$\kappa_{(q+1)^*}' \geq \kappa_{(q+1)^*}^2 + 4\lambda, \tag{8.60}$$

$$\kappa_i' \geq \kappa_i^2 + \lambda \qquad (i = q+2, \ldots, n^*), \tag{8.61}$$

together with the initial conditions:

$$\kappa_a(0) \quad \text{finite} \quad \text{for } a = 1, \ldots, q^*, \tag{8.62}$$

$$\kappa_i(0) \quad = -\infty \quad \text{for } i = (q+1)^*, \ldots, n^*. \tag{8.63}$$

For (8.59) and (8.61) we have used the assumption that $K_{\mathrm{ah}}^M \geq \lambda$, and for (8.60) we have used the assumption that $K_{\mathrm{hol}}^M \geq 4\lambda$. Then (8.59)–(8.63) can be solved using Lemmas 8.18 and 8.19. The result is:

$$\begin{cases} \kappa_a(t) \geq -\dfrac{d}{dt}\log\left(\cos(t\sqrt{\lambda}) - \dfrac{\kappa_a(0)}{\sqrt{\lambda}}\sin(t\sqrt{\lambda})\right), \\[2ex] \kappa_{(q+1)^*} \geq \dfrac{-2\sqrt{\lambda}}{\tan(2t\sqrt{\lambda})} = -\dfrac{d}{dt}\log(\sin(2t\sqrt{\lambda})), \\[2ex] \kappa_i(t) \geq -\dfrac{d}{dt}\log(\sin(t\sqrt{\lambda})), \end{cases} \tag{8.64}$$

for $a = 1, \ldots, q^*$ and $i = q+2, \ldots, n^*$. We also get that (8.40) holds. Summing the principal curvature functions, we find that

$$\begin{aligned} \operatorname{tr} S(t) \geq & -\frac{d}{dt}\Bigg\{\sum_{a=1}^{q^*}\log\left(\cos(t\sqrt{\lambda}) - \frac{\kappa_a(0)}{\sqrt{\lambda}}\sin(t\sqrt{\lambda})\right) \\ & +2(n-q-1)\log\sin(t\sqrt{\lambda}) + \log\sin(2t\sqrt{\lambda})\Bigg\} \\ = & -\frac{d}{dt}\log\Bigg\{(\sin(t\sqrt{\lambda}))^{2n-2q-1}\cos(t\sqrt{\lambda}) \\ & \cdot\det\left(\cos(t\sqrt{\lambda})I - \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}T_u\right)\Bigg\}. \end{aligned} \tag{8.65}$$

Then (8.65) is equivalent to (8.58). □

Similarly, corresponding to Lemma 8.21 we have

Lemma 8.32. *Suppose the holomorphic and antiholomorphic sectional curvatures satisfy* $K_{\mathrm{hol}}^M \leq 4\lambda$ *and* $K_{\mathrm{ah}}^M \leq \lambda$. *Assume that* (8.38) *and* (8.40) *hold, and that along each normal geodesic* ξ *to* P *the vector field* $J\xi'(t)$ *is an eigenvector of* $S(t)$. *Then on* $(0, e_c(p,u))$ *the inequality in* (8.58) *is reversed.*

The analog of Lemma 8.22 is:

Lemma 8.33. *Suppose* $\lambda \geq 0$. *If* $K_{\mathrm{hol}}^M \geq 4\lambda$ *and* $K_{\mathrm{ah}}^M \geq \lambda$, *then for* $0 \leq t \leq e_c(p,u)$

$$t \longmapsto \Theta_u(t)\left\{\left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{2n-2q-1} \cos(t\sqrt{\lambda}) \det\left(\cos(t\sqrt{\lambda})I - \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}T_u\right)\right\}^{-1}$$

is a nonincreasing function, and

$$\begin{aligned}\Theta_u(t) &\leq \left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{2n-2q-1} \cos(t\sqrt{\lambda}) \det\left(\cos(t\sqrt{\lambda})I - \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}T_u\right) \\ &\leq \left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{2n-2q-1} (\cos(t\sqrt{\lambda}))^{2q+1}.\end{aligned} \tag{8.66}$$

Proof. All is clear except perhaps for the second inequality of (8.66). In fact, using the standard inequality between the arithmetic and geometric mean we have

$$\begin{aligned}&\det\left(\cos(t\sqrt{\lambda})I - \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}T_u\right) \\ &\qquad = (\cos(t\sqrt{\lambda}))^{2q} \det\left(I - \frac{\tan(t\sqrt{\lambda})}{\sqrt{\lambda}}T_u\right) \\ &\qquad \leq (\cos(t\sqrt{\lambda}))^{2q}\left(1 - \frac{\tan(t\sqrt{\lambda})}{2q\sqrt{\lambda}}\langle H, u\rangle\right)^{2q}.\end{aligned}$$

But since P is a Kähler submanifold of M, the mean curvature vector field H vanishes identically (see Lemma 6.26). Thus we get (8.66). □

Similarly, the analog of Lemma 8.24 is:

Lemma 8.34. *Assume the hypotheses of Lemma* 8.32. *Then for* $0 \leq t < e_c(p,u)$

$$t \longmapsto \Theta_u(t)\left\{\left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{2n-2q-1} \cos(t\sqrt{\lambda}) \det\left(\cos(t\sqrt{\lambda})I - \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}T_u\right)\right\}^{-1}$$

is a nondecreasing function, and

$$\Theta_u(t) \geq \left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{2n-2q-1} \cos(t\sqrt{\lambda}) \det\left(\cos(t\sqrt{\lambda})I - \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}T_u\right).$$

We are at last able to give the Kähler analogs of Theorems 8.16 and 8.17. They also generalize Theorem 7.19. Recall that $\gamma(R^P - R^M)$ denotes the total Chern form of the curvature tensor $R^P - R^M$.

Theorem 8.35. *Suppose $K^M_{\mathrm{hol}} \geq 4\lambda$ and $K^M_{\mathrm{ah}} \geq \lambda$. Then for $r \leq \mathrm{minfoc}(P)$ we have*

$$\begin{aligned} V^M_P(r) &\leq \frac{1}{n!}\int_P \gamma(R^P - R^M) \wedge \frac{\left(\frac{\pi}{\lambda}\sin^2(r\sqrt{\lambda}) + \cos^2(r\sqrt{\lambda})F\right)^n}{1 - \frac{\lambda}{\pi}F} \\ &\leq \frac{1}{(n-q)!}\left(\frac{\pi}{\lambda}\sin^2(r\sqrt{\lambda})\right)^{n-q} \mathrm{volume}(P). \end{aligned} \tag{8.67}$$

Theorem 8.36. *If $K^M_{\mathrm{hol}} \leq 4\lambda$ and $K^M_{\mathrm{ah}} \leq \lambda$, then for $0 \leq r \leq \mathrm{minfoc}(P)$ we have*

$$V^M_P(r) \geq \frac{1}{n!}\int_P \gamma(R^P - R^M) \wedge \frac{\left(\frac{\pi}{\lambda}\sin^2(r\sqrt{\lambda}) + \cos^2(r\sqrt{\lambda})F\right)^n}{1 - \frac{\lambda}{\pi}F}.$$

Important special cases of Theorems 8.35 and 8.36 are:

Corollary 8.37. *Let P be a Kähler submanifold of M and suppose $K^M \geq 0$. Then for $r \leq \mathrm{minfoc}(P)$*

$$\begin{aligned} V^M_P(r) &\leq \frac{1}{n!}\int_P \gamma(R^P - R^M) \wedge (\pi r^2 + F)^n \\ &\leq \frac{(\pi r^2)^{(n-q)}}{(n-q)!}\mathrm{volume}(P). \end{aligned}$$

Corollary 8.38. *Suppose $K^M \leq 0$. Then for $0 \leq r \leq \mathrm{minfoc}(P)$ we have*

$$V^M_P(r) \geq \frac{1}{n!}\int_P \gamma(R^P - R^M) \wedge (\pi r^2 + F)^n.$$

Note that Corollaries 8.37 and 8.38 also follow from Theorem 8.4 and equation 7.15.

Finally, as a special case of Theorem 8.35 we have a generalization of Lemma 6.18 that can be considered a Kähler analog of the Bishop-Günther Inequalities:

Corollary 8.39. *Let M be a complete Kähler manifold.*

(i) *Suppose* $K_{\mathrm{hol}}^M \geq 4\lambda$ *and* $K_{\mathrm{ah}}^M \geq \lambda$. *Then for all* $m \in M$ *and* $r \geq 0$

$$V_m^M(r) \leq \frac{1}{n!}\left(\frac{\pi}{\lambda}\right)^n \left(\sin(r\sqrt{\lambda})\right)^{2n}. \tag{8.68}$$

(ii) *Suppose that* $r \geq 0$ *is less than or equal to the distance from* m *to its nearest cut point. If* $K_{\mathrm{hol}}^M \leq 4\lambda$ *and* $K_{\mathrm{ah}}^M \leq \lambda$, *then the inequality in* (8.68) *is reversed.*

8.6 Some Inequalities of Heintze and Karcher

In Section 3.6 we showed that for a compact Riemannian manifold M with $K^M \geq \lambda$ the inequality

$$\mathrm{volume}(M) \leq \mathrm{volume}(S^n(\lambda))$$

holds. A key ingredient in the proof of this theorem was the fact that the infinitesimal change of volume function of M at any point was less than or equal to the corresponding infinitesimal change of volume function for a point in a sphere $S^n(\lambda)$. We have already seen (equation (8.41)) that there is a similar inequality between the infinitesimal change of volume functions of a submanifold in M and a submanifold in $S^n(\lambda)$; we used (8.41) to estimate tube volumes in Section 8.3.

Heintze and Karcher [HK] have given an estimate for the volume of a compact Riemannian manifold with $K^M \geq \lambda$ in terms of the volume of a submanifold P of M. This estimate is a consequence of the second inequality of (8.41), which they prove using Jacobi field techniques. Moreover, M can be thought of as sort of generalized tube about P in which the radius of the tube is allowed to change from point to point. In this section we modify the techniques of Section 8.3 in order to prove the inequality of Heintze and Karcher. Actually, we sharpen their inequality a little, because the first inequality of (8.41) is also available. There are several related inequalities in [HK] that will not be discussed here.

First, we note a general formula relating the volume of a Riemannian manifold to one of its compact submanifolds. It looks very much like formula (8.4) and explains why we can think of a compact manifold M as a tube of variable radius about a compact submanifold P.

Theorem 8.40. *Let* P *be a* q*-dimensional compact submanifold of a compact* n*-dimensional Riemannian manifold* M, $q < n-1$. *Then*

$$\begin{aligned}\mathrm{volume}(M) &= \mathrm{volume}(\mathcal{O}_P) \\ &= \int_P \int_{S^{n-q-1}(1)} \int_0^{e_c(p,u)} t^{n-q-1}\Theta_u(t)\,dt\,du\,dP.\end{aligned} \tag{8.69}$$

Proof. Since both P and M are compact, by the Hopf-Rinow Theorem (see for example [BC, page 154]) any point $m \in M$ has a shortest geodesic from it to P. This geodesic meets P orthogonally. It follows that M is the closure of $\mathcal{O}_P$, and

therefore they have the same volume. To compute the volume of $\mathcal{O}_P$, we proceed as in the proof of Lemma 8.2. For each $(p,u) \in \boldsymbol{\nu}$ with $\|u\| = 1$ there is a unique unit-speed geodesic ξ emanating from p with $\xi'(0) = u$. Moreover, all such geodesics fill up $\mathcal{O}_P$. Therefore, when we transfer the integration from M to $\boldsymbol{\nu}$ as we did in Lemma 8.2 we get (8.69). □

We specialize (8.69) to a case we shall need:

Corollary 8.41. *Let the sphere $S^q(\lambda + \Lambda^2/q^2)$ be embedded as a sphere of latitude in a sphere $S^n(\lambda)$, and denote by H the mean curvature vector of the embedding. Then*

$$\begin{aligned}&\operatorname{volume}(S^n(\lambda)) \\ &= \operatorname{volume}\left(S^q\left(\lambda + \frac{\Lambda^2}{q^2}\right)\right) \int_{S^{n-q-1}(1)} \int_0^{x(u)} \left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-q-1} \\ &\qquad \cdot \left(\cos(t\sqrt{\lambda}) - \frac{\sin(t\sqrt{\lambda})}{q\sqrt{\lambda}}\langle H, u\rangle\right)^q dt\, du,\end{aligned} \tag{8.70}$$

where $x(u)$ is defined by

$$\cos(x(u)\sqrt{\lambda}) = \frac{\sin(x(u)\sqrt{\lambda})}{\sqrt{\lambda}}\langle H, u\rangle.$$

Proof. Let T_u be the Weingarten map of the embedding of the sphere $S^q(\lambda + \Lambda^2/q^2)$ into the sphere $S^n(\lambda)$. Since a sphere of latitude is totally umbilic, we have

$$T_u = \frac{1}{q}\langle H, u\rangle I.$$

So, the radius of the sphere of latitude is

$$\frac{1}{\sqrt{\lambda + \|H\|^2/q^2}} = \frac{1}{\sqrt{\lambda + \Lambda^2/q^2}},$$

from which we conclude that $\|H\|^2 = \Lambda^2$. Also, it follows from Corollary 8.26 that the infinitesimal change of volume function is given by

$$\vartheta_u(t) = \left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-q-1} \left(\cos(t\sqrt{\lambda}) - \frac{\sin(t\sqrt{\lambda})}{q\sqrt{\lambda}}\langle H, u\rangle\right)^q.$$

Then $e_c(p,u)$ is the first zero of $\vartheta_u(t)$ along $t \longmapsto \exp_\nu(p, tu)$. This is just $x(u)$, so $x(u) = e_c(p,u)$. Since both $\vartheta_u(t)$ and $x(u)$ are independent of the point p, we can reduce the triple integral on the right-hand side of (8.69) to a double integral. Thus we get (8.70). □

Corollary 8.42. *Let $q > 0$. Then for any compact q-dimensional submanifold P of a compact Riemannian manifold M we have*

$$\int_{S^{n-q-1}(1)} \int_0^{x(u)} \left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-q-1} \left(\cos(t\sqrt{\lambda}) - \frac{\sin(t\sqrt{\lambda})}{q\sqrt{\lambda}}\langle H, u\rangle\right)^q dt\, du \tag{8.71}$$
$$= \frac{\mathrm{volume}(S^n(\lambda))}{\mathrm{volume}\left(S^q\left(\lambda + \dfrac{\|H\|^2}{q^2}\right)\right)}$$
$$= \frac{\Gamma\big(\frac{1}{2}(q+1)\big)\, \pi^{(n-q)/2} \left(\lambda + \dfrac{\|H\|^2}{q^2}\right)^{q/2}}{\Gamma\big(\frac{1}{2}(n+1)\big)\, \lambda^{n/2}}.$$

Proof. The calculation of the right-hand side of (8.70) is a pointwise computation, so it is the same for all manifolds. Therefore, it can be found by doing the special case of an immersion of $S^q(\lambda + \|H\|^2/q^2)$ as a sphere of latitude in a sphere $S^n(\lambda)$. Thus we get (8.71) from (8.70). $\square$

Now we prove the main result of this section.

Theorem 8.43. *Let P be a compact q-dimensional submanifold of a compact n-dimensional Riemannian manifold M whose sectional curvature satisfies $K^M \geq \lambda$. Then*

$$\mathrm{volume}(M) \leq \int_P \int_{S^{n-q-1}(1)} \int_0^{x(u)} \left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-q-1} \tag{8.72}$$
$$\cdot \det\left(\cos(t\sqrt{\lambda})I - \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}} T_u\right) dt\, du\, dP$$
$$\leq \int_P \int_{S^{n-q-1}(1)} \int_0^{x(u)} \left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{n-q-1}$$
$$\cdot \left(\cos(t\sqrt{\lambda}) - \frac{\sin(t\sqrt{\lambda})}{q\sqrt{\lambda}}\langle H, u\rangle\right)^q dt\, du\, dP.$$

Proof. From (8.69) and (8.41) we get

$$\mathrm{volume}(M) \leq \int_P \int_{S^{n-q-1}(1)} \int_0^{e_c(p,u)} \left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-q-1} \tag{8.73}$$
$$\cdot \det\left(\cos(t\sqrt{\lambda})I - \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}} T_u\right) dt\, du\, dP$$

$$\leq \int_P \int_{S^{n-q-1}(1)} \int_0^{e_c(p,u)} \left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}} \right)^{n-q-1} \cdot \left(\cos(t\sqrt{\lambda}) - \frac{\sin(t\sqrt{\lambda})}{q\sqrt{\lambda}} \langle H, u \rangle \right)^q dt\, du\, dP.$$

The integrands on the right-hand side of (8.73) remain nonnegative when we replace $e_c(p,u)$ by $x(u)$, and so we get (8.72). □

From Theorem 8.43 and Corollary 8.42 we get a corollary.

Corollary 8.44. *Under the hypotheses of Theorem 8.43 we have*

$$\begin{aligned} \operatorname{volume}(M) &\leq \operatorname{volume}(S^n(\lambda)) \int_P \frac{dP}{\operatorname{volume}\left(S^q \left(\lambda + \frac{\|H\|^2}{q^2} \right) \right)} \\ &= \frac{\pi^{(n-q)/2}\Gamma(\frac{1}{2}(q+1))}{\Gamma(\frac{1}{2}(n+1))\lambda^{n/2}} \int_P \left(\lambda + \frac{\|H\|^2}{q^2} \right)^{q/2} dP. \end{aligned}$$

If in addition $\|H\| \leq \Lambda$, *then*

$$\frac{\operatorname{volume}(M)}{\operatorname{volume}(S^n(\lambda))} \leq \frac{\operatorname{volume}(P)}{\operatorname{volume}\left(S^q \left(\lambda + \frac{\Lambda^2}{q^2} \right) \right)}.$$

Related inequalities for Kähler manifolds can be found in [Gim], [GM] and [MP2].

8.7 Gromov's Improvement of the Bishop-Günther Inequalities

In this section the submanifold P will always be a point m. We computed $V_m^{\mathbb{K}^n(\lambda)}(r)$ explicitly in Corollary 3.18; its derivative $A_m^{\mathbb{K}^n(\lambda)}(r)$ is given by

$$A_m^{\mathbb{K}^n(\lambda)}(r) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}} \right)^{n-1}. \tag{8.74}$$

Gromov [Gromov] has given an argument that improves the Bishop-Günther Inequalities in two ways. Suppose that $\lambda \geq 0$, and that M is a complete manifold whose Ricci curvature is bounded below by λ. Then not only is it true that

$$V_m^M(r) \leq V_m^{\mathbb{K}^n(\lambda)}(r), \tag{8.75}$$

but also that

$$r \longmapsto \frac{\mathrm{V}_m^M(r)}{\mathrm{V}_m^{\mathbb{K}^n(\lambda)}(r)} \tag{8.76}$$

is nonincreasing. In fact, (8.75) follows from (8.76) and the fact that

$$\lim_{r\to 0} \frac{\mathrm{V}_m^M(r)}{\mathrm{V}_m^{\mathbb{K}^n(\lambda)}(r)} = 1.$$

Thus the question arises whether or not (8.75) can be replaced by the stronger statement (8.76).

Also, in Theorem 3.19 it was assumed that r was less than the distance from m to its nearest cut point. Since both $V_m^M(r)$ and $V_m^{\mathbb{K}^n(\lambda)}(r)$ make sense for arbitrary r, a second question is whether or not (8.75) and (8.76) continue to hold for large r. Both questions were answered affirmatively by Gromov [Gromov, page 65]. We can now give a simple proof of his theorem.

Theorem 8.45. *If the Ricci curvature of M satisfies*

$$\rho^M(x,x) \ \geq \ (n-1)\lambda\|x\|^2 \tag{8.77}$$

for all tangent vectors x to M, then for all $r \geq 0$

$$r \longmapsto \frac{\mathrm{V}_m^M(r)}{\mathrm{V}_m^{\mathbb{K}^n(\lambda)}(r)} \tag{8.78}$$

is nonincreasing.

Proof. From Lemma 8.28 we know that

$$t \longmapsto \frac{\Theta_u(t)}{\max\left(\left(\dfrac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{n-1}, 0\right)}. \tag{8.79}$$

is nonincreasing for all $t \geq 0$. Consequently,

$$\Theta_u(t)\left(\frac{\sin(s\sqrt{\lambda})}{s\sqrt{\lambda}}\right)^{n-1} \leq \Theta_u(s)\left(\frac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{n-1} \tag{8.80}$$

Integrating (8.80) over the unit sphere in M_m and using (8.74), we obtain

$$\mathrm{A}_m^M(t)\mathrm{A}_m^{\mathbb{K}^n(\lambda)}(s) \leq \mathrm{A}_m^M(s)\mathrm{A}_m^{\mathbb{K}^n(\lambda)}(t). \tag{8.81}$$

Then integrating (8.81) with respect to s from 0 to t, we get

$$\mathrm{A}_m^M(t)\mathrm{V}_m^{\mathbb{K}^n(\lambda)}(t) \le \mathrm{V}_m^M(t)\mathrm{A}_m^{\mathbb{K}^n(\lambda)}(t). \tag{8.82}$$

But (8.82) implies that

$$\frac{d}{dt}\left(\frac{\mathrm{V}_m^M(t)}{\mathrm{V}_m^{\mathbb{K}^n(\lambda)}(t)}\right) = \frac{\mathrm{A}_m^M(t)\mathrm{V}_m^{\mathbb{K}^n(\lambda)}(t) - \mathrm{V}_m^M(t)\mathrm{A}_m^{\mathbb{K}^n(\lambda)}(t)}{\left(\mathrm{V}_m^{\mathbb{K}^n(\lambda)}(t)\right)^2} \le 0,$$

and so we get (8.78). □

There is a Kähler analog of Theorem 8.45 (see also [Nay]). Let $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ denote a space of constant holomorphic sectional curvature 4λ.

Theorem 8.46. *Let M be a complete Kähler manifold.*

(i) *If $K^M_{\mathrm{hol}} \ge 4\lambda$ and $K^M_{\mathrm{ah}} \ge \lambda$, then*

$$r \longmapsto \frac{\mathrm{V}_m^M(r)}{\mathrm{V}_m^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}(r)} \tag{8.83}$$

is nonincreasing for all $r \ge 0$.

(ii) *If $K^M_{\mathrm{hol}} \le 4\lambda$ and $K^M_{\mathrm{ah}} \le \lambda$, then*

$$r \longmapsto \frac{\mathrm{V}_m^M(r)}{\mathrm{V}_m^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}(r)} \tag{8.84}$$

is nondecreasing for $0 \le r \le e_c(m)$ for all $m \in M$.

Proof. For a Kähler manifold M with $K^M_{\mathrm{hol}} \ge 4\lambda$ and $K^M_{\mathrm{ah}} \ge \lambda$ we have as a special case of Lemma 8.33 that

$$t \longmapsto \frac{\Theta_u(t)}{\max\left(\left(\dfrac{\sin(t\sqrt{\lambda})}{t\sqrt{\lambda}}\right)^{2n-1}, 0\right)\cos(t\sqrt{\lambda})} \tag{8.85}$$

is nonincreasing for $0 \le t < \dfrac{\pi}{2\sqrt{\lambda}}$. The rest of the proof of (i) is the same as that of Theorem 8.45, but using (8.85) instead of (8.79). The proof of (ii) is similar. □

8.8 Ball and Tube Comparison Theorems for Surfaces

In view of Theorems 8.4 and 8.16 the following conjecture naturally arises:

Conjecture. *Let M and $\widetilde{M}$ be Riemannian manifolds such that $K^M \geq K^{\widetilde{M}}$ and let $\phi: M \longrightarrow \widetilde{M}$ be a diffeomorphism. Then $V_P^M(r) \leq V_{\widetilde{P}}^{\widetilde{M}}(r)$, where P and $\widetilde{P}$ are submanifolds of M and $\widetilde{M}$ such that $\phi(P) = \widetilde{P}$.*

Before trying to resolve this conjecture we should decide what the condition

$$\text{“}K^M \geq K^{\widetilde{M}}\text{”} \tag{8.86}$$

really means. In fact, there is more than one way to interpret (8.86). For example, we could assume

$$\inf_{m\in M} K_m^M \geq \sup_{\widetilde{m}\in\widetilde{M}} K_{\widetilde{m}}^{\widetilde{M}}. \tag{8.87}$$

An interpretation of (8.86) weaker than (8.87) is

$$K^M(\Pi) \geq K^{\widetilde{M}}(\widetilde{\Pi}), \tag{8.88}$$

whenever Π and $\widetilde{\Pi}$ are 2-dimensional subspaces of tangent spaces such that $\phi_*(\Pi) = \widetilde{\Pi}$. In this section we shall show that the conjecture with the hypothesis $K^M \geq K^{\widetilde{M}}$ interpreted as (8.88) holds for geodesic balls in surfaces. For a different approach that uses Jacobi fields and is sometimes more general, see [HK] and [BZ, page 244].

Let M and $\widetilde{M}$ be complete 2-dimensional Riemannian manifolds. Let $m \in M$ and $\widetilde{m} \in \widetilde{M}$, and suppose that $\Phi: M_m \longrightarrow \widetilde{M}_{\widetilde{m}}$ is an isometry. We put

$$\phi = \exp_{\widetilde{m}} \circ \Phi \circ \exp_m^{-1}. \tag{8.89}$$

Then ϕ is defined in the neighborhood $\mathcal{O}_m$ of $m \in M$. This neighborhood is the complement of the cut locus $\mathrm{Cut}(m)$ of m. Similarly, let $\widetilde{\mathcal{O}}_{\widetilde{m}}$ denote the complement of the cut locus $\mathrm{Cut}(\widetilde{m})$. Denote by $\vartheta_u(t)$ and $\widetilde{\vartheta}_{\widetilde{u}}(t)$ the infinitesimal change of volume functions of M and $\widetilde{M}$. The distance from m to its nearest cut point along the geodesic $t \longmapsto \exp_m(tu)$ will be denoted by $e_c(m,u)$. Let $e_c(\widetilde{m},\widetilde{u})$ denote the corresponding function for $\widetilde{M}$, where $\widetilde{u} = \Phi(u)$. The sectional curvatures of M and $\widetilde{M}$ will be denoted by K and $\widetilde{K}$, respectively. Then we put $K_u(t) = K(\exp_m(tu))$ and similarly for $\widetilde{M}$.

The following lemma uses techniques of M. Bôcher[2] [Bôcher1], [Bôcher2], [Bôcher3] (see especially the first footnote on page 435 of [Bôcher2]):

[2]Maxime Bôcher(1867–1918). American mathematician. After studying at Harvard, Bôcher went to Göttingen as a visiting fellow, where he wrote a book **Über die Reihenentwickelungen der Potentialtheorie**, which also served as his doctoral dissertation. He returned to Harvard, where he became one of the first editors of the *Annals of Mathematics* and was elected a member of the National Academy of Sciences. Bôcher was a prolific contributor to mathematical journals on differential equations and potential theory; he also wrote several text books.

Lemma 8.47. *For $0 \le t \le e_c(m,u)$ we have*

$$\log\left(\frac{\vartheta_u(t)}{\widetilde{\vartheta}_{\tilde{u}}(t)}\right) = \int_0^t \int_0^s \frac{x^2\vartheta_u(x)\widetilde{\vartheta}_{\tilde{u}}(x)}{s^2\vartheta_u(s)\widetilde{\vartheta}_{\tilde{u}}(s)} \left(\widetilde{K}_{\tilde{u}}(x) - K_u(x)\right) dx\, ds. \tag{8.90}$$

Proof. Let S denote the shape operator for the geodesic balls about $m \in M$, and let $\widetilde{S}$ denote the corresponding operator in $\widetilde{M}$. Put

$$F(x) = \log\left(\frac{\vartheta_u(x)}{\widetilde{\vartheta}_{\tilde{u}}(x)}\right) \qquad \text{and} \qquad G(x) = -\log\bigl(x^2\vartheta_u(x)\widetilde{\vartheta}_{\tilde{u}}(x)\bigr).$$

From Theorem 3.11 it follows that

$$F''(x) = F'(x)G'(x) + \widetilde{K}_{\tilde{u}}(x) - K_u(x). \tag{8.91}$$

Since $\vartheta_u(0) = \widetilde{\vartheta}_{\tilde{u}}(0) = 1$ and $\vartheta_u'(0) = \widetilde{\vartheta}_{\tilde{u}}'(0) = 0$, we have the initial conditions

$$F'(0) = (e^{-G})(0) \;=\; 0. \tag{8.92}$$

Equation (8.91) can be rewritten as

$$\left(e^{-G(x)}F'(x)\right)' = e^{-G(x)}\left(\widetilde{K}_{\tilde{u}}(x) - K_u(x)\right). \tag{8.93}$$

When (8.93) is integrated from 0 to s and the initial conditions (8.92) are used, we get

$$\begin{aligned} F'(s) &= \int_0^s e^{G(s)-G(x)}\left(\widetilde{K}_{\tilde{u}}(x) - K_u(x)\right) dx \qquad (8.94)\\ &= \int_0^s \frac{x^2\vartheta_u(x)\widetilde{\vartheta}_{\tilde{u}}(x)\left(\widetilde{K}_{\tilde{u}}(x) - K_u(x)\right)}{s^2\vartheta_u(s)\widetilde{\vartheta}_{\tilde{u}}(s)} dx. \end{aligned}$$

Then (8.90) follows upon integrating (8.94) from 0 to t and using the fact that $\vartheta_u(0) = \widetilde{\vartheta}_{\tilde{u}}(0) = 1$. □

Theorem 8.48. *Suppose that for all $m \in M$ the sectional curvatures K and $\widetilde{K}$ of surfaces M of $\widetilde{M}$ satisfy*

$$K \ge \widetilde{K} \circ \phi \tag{8.95}$$

for some isometry $\Phi\colon M_m \longrightarrow \widetilde{M}_{\tilde{m}}$, where ϕ and Φ are related by (8.89). Then:

(i) *for $0 \le t \le e_c(m,u)$ the function $t \longmapsto \dfrac{\vartheta_u(t)}{\widetilde{\vartheta}_{\tilde{u}}(t)}$ is nonincreasing;*

(ii) $\vartheta_u(t) \le \widetilde{\vartheta}_{\tilde{u}}(t)$ *for $0 \le t \le e_c(m,u)$;*

(iii) $e_c(m,u) \le \widetilde{e}_c(\widetilde{m},\widetilde{u})$;

(iv) *if M and $\widetilde{M}$ are compact and simply connected, then*

$$\operatorname{volume}(M) \le \operatorname{volume}(\widetilde{M}).$$

Proof. From (8.90) and (8.95) it follows that

$$\frac{d}{dt}\log\left(\frac{\vartheta_u(t)}{\widetilde{\vartheta}_{\widetilde{u}}(t)}\right) \le 0,$$

and from this part (i) is obvious. Then (i) implies that

$$0 \le \frac{\vartheta_u(t)}{\widetilde{\vartheta}_{\widetilde{u}}(t)} \le \frac{\vartheta_u(0)}{\widetilde{\vartheta}_{\widetilde{u}}(0)} = 1,$$

and so (ii) and (iii) follow.

The proof of (iv) is similar to that of (ii) and Theorem 3.22, page 49, but instead of integrating over a geodesic ball, we integrate over the star-like regions $\mathcal{O}_m$ and $\widetilde{\mathcal{O}}_{\widetilde{m}}$:

$$\begin{aligned}
\operatorname{volume}(M) &= \int_{\mathcal{O}_m} t\,\vartheta_u(t)\,du\,dt\\
&= \int_{\|u\|=1}\int_0^{e_c(m,u)} t\,\vartheta_u(t)\,dt\,du\\
&\le \int_{\|\widetilde{u}\|=1}\int_0^{\widetilde{e}_c(\widetilde{m},\widetilde{u})} t\widetilde{\vartheta}_{\widetilde{u}}(t)\,dt\,d\widetilde{u} = \operatorname{volume}(\widetilde{M}).
\end{aligned}$$ □

Part (iii) of Theorem 8.48 can be improved provided that the hypothesis (8.95) is strengthened. We use the method of proof of Theorem 8.45. Thus in the case of surfaces we also obtain a sharpened version of Gromov's Theorem.

Theorem 8.49. *Suppose the sectional curvatures of surfaces M and $\widetilde{M}$ satisfy* (8.95) *for any choice of linear isometry $\Phi\colon M_m \longrightarrow \widetilde{M}_{\widetilde{m}}$. Then*

$$r \longmapsto \frac{\mathrm{V}_m^M(r)}{\mathrm{V}_{\widetilde{m}}^{\widetilde{M}}(r)} \tag{8.96}$$

is nonincreasing.

Proof. From part (i) of Theorem 8.48 it follows that

$$\vartheta_u(t)\widetilde{\vartheta}_{\widetilde{u}}(s) \le \vartheta_u(s)\widetilde{\vartheta}_{\widetilde{u}}(t) \tag{8.97}$$

for $0 \le s \le t \le e_c(m,u)$. Because we are assuming that (8.86) holds for all choices of the isometry between M_m and $\widetilde{M}_{\widetilde{m}}$, it follows that (8.97) holds for all choices of u and $\widetilde{u}$ with $\|u\| = \|\widetilde{u}\| = 1$. We have

$$\mathrm{A}_m^M(t) = \int_{\|u\|=1} t\,\vartheta_u(t)\,du,$$

and similarly for $A_{\widetilde{m}}^{\widetilde{M}}(t)$. When we integrate (8.97) over the unit spheres in M_m and $\widetilde{M}_{\widetilde{m}}$, we get

$$\mathrm{A}_m^M(t)\mathrm{A}_{\widetilde{m}}^{\widetilde{M}}(s) \le \mathrm{A}_m^M(s)\mathrm{A}_{\widetilde{m}}^{\widetilde{M}}(t). \tag{8.98}$$

We integrate (8.98) with respect to s from 0 to t and obtain

$$\mathrm{A}_m^M(t)\mathrm{V}_{\widetilde{m}}^{\widetilde{M}}(t) \le \mathrm{V}_m^M(t)\mathrm{A}_{\widetilde{m}}^{\widetilde{M}}(t). \tag{8.99}$$

On the other hand, we have

$$\frac{d}{dt}\left(\frac{\mathrm{V}_m^M(t)}{\mathrm{V}_{\widetilde{m}}^{\widetilde{M}}(t)}\right) = \frac{\mathrm{A}_m^M(t)\mathrm{V}_{\widetilde{m}}^{\widetilde{M}}(t) - \mathrm{V}_m^M(t)\mathrm{A}_{\widetilde{m}}^{\widetilde{M}}(t)}{\left(\mathrm{V}_{\widetilde{m}}^{\widetilde{M}}(t)\right)^2}. \tag{8.100}$$

From (8.99) and (8.100) follows (8.96). □

8.9 Comparison Theorems for Riemannian Manifolds

For general dimensions, the proof of the conjecture in Section 8.8 needs a different method. We follow here a simple one developed by J. Eschenburg and E. Heintze [EH] which still makes use of the shape operator $S(t)$. For different approaches, see [Esch] and [Royden].

The key lemma in this proof will be 8.51. In order to prove it, we shall need the following elementary fact: Let V be an euclidean vector space of dimension n. We shall denote by $\mathcal{S}(V)$ the set of symmetric endomorphisms of V (symmetric respect to the scalar product $\langle\,,\rangle$ in V).

Lemma 8.50. *Let $X\colon (0,t_0) \to \mathcal{S}(V)$ be a continuous function. Then there is a solution $g\colon (0,t_0) \to \mathcal{S}(V)$ of the differential equation $g' = Xg$ which is nonsingular at every $t \in (0,t_0)$.*

Proof. Take $g(t)$ as the fundamental matrix associated to the system of ordinary differential equations

$$v'(t) = X(t)\,v(t) \qquad (\text{with } v\colon (0,t_0) \to V)$$

and satisfying $g(t_1) = \mathrm{Id}$ for some $t_1 \in (0, t_0)$ (cf. [GMP, page 510]). Let $\bar{g}$ be the solution of the equation

$$\bar{g}' = -\bar{g}X, \qquad \text{satisfying } \bar{g}(t_1) = \mathrm{Id}.$$

Then $(g\bar{g})' = g'\bar{g} + g\bar{g}' = Xg\bar{g} - g\bar{g}X = 0$, and $g\bar{g}(t_1) = \mathrm{Id}$, then $g\bar{g}(t) = \mathrm{Id}$ for every $t \in (0, t_0)$ and g and $\bar{g}$ are nonsingular at every $t \in (0, t_0)$. □

Lemma 8.51. *Let $R_1, R_2 \colon \mathbb{R} \to \mathcal{S}(V)$ be C^∞ maps satisfying $R_1 \geq R_2$ (that is, $\langle R_1 u, u\rangle \geq \langle R_2 u, u\rangle$ for every $u \in V$). If $S_i \colon (0, t_i) \to \mathcal{S}(V)$ are solutions of*

$$S_i' = S_i^2 + R_i, \tag{8.101}$$

with maximal $t_i \in (0, \infty)$, $U(t) = S_1(t) - S_2(t)$, and there exists $U(0) := \lim_{t\to 0} U(t) \geq 0$, then $t_1 \leq t_2$ and $S_1 \geq S_2$ on $(0, t_1)$.

Proof. First, let us remark that

$$2(S_1^2 - S_2^2) = (S_1 + S_2)U + U(S_1 + S_2). \tag{8.102}$$

Let $t_0 = \min\{t_1, t_2\}$. By (8.101) and (8.102), U satisfies

$$\begin{aligned} U' = S_1' - S_2' &= S_1^2 + R_1 - S_2^2 - R_2 = S_1^2 - S_2^2 + (R_1 - R_2) \\ &= XU + UX + (R_1 - R_2), \end{aligned} \tag{8.103}$$

where $X = \frac{1}{2}(S_1 + S_2)$.

Let $g \colon (0, t_0) \to \mathcal{S}(V)$ be a nonsingular solution of $g' = Xg$. Define W by

$$U = gWg^t. \tag{8.104}$$

Then

$$U' = g'Wg^t + gW'g^t + gWg^{t'} = XgWg^t + gW'g^t + gWg^tX^t = XU + gW'g^t + UX^t.$$

But, since X is symmetric, we have that (8.103) holds if and only if

$$W' = g^{-1}(R_1 - R_2)(g^{-1})^t. \tag{8.105}$$

Then $R_1 - R_2 \geq 0$ implies, for every $u \in V$

$$\langle W'u, u\rangle = \langle (R_1 - R_2)(g^{-1})^t u, (g^{-1})^t u\rangle \geq 0. \tag{8.106}$$

From (8.104), it is possible to see (cf. [EH]) that $W(0) := \lim_{t\to 0} W(t)$ exists and, since $U(0) \geq 0$, $W(0) \geq 0$. From this and (8.106) it follows that $W(t) \geq 0$ and $U(t) \geq 0$ for every $t \in (0, t_0)$. From (8.101) it follows that

$$\langle S_i' u, u\rangle = |S_i u|^2 + \langle R_i u, u\rangle$$

is bounded from below, and therefore, $\langle S_i u, u\rangle$ tends to $+\infty$ at a singularity. Since $\langle S_1(t)u, u\rangle \geq \langle S_2(t)u, u\rangle$ for $t < t_0 = \min\{t_1, t_2\}$, it follows that $t_0 = t_1 \leq t_2$. □

Theorem 8.52. *Let M and $\tilde{M}$ be Riemannian manifolds, $m \in M$, $\tilde{m} \in \tilde{M}$ and ϕ defined by* (8.89). *If* (8.88) *is satisfied, then:*

(i) *for* $0 \le t \le e_c(m,u)$, *the function* $t \mapsto \frac{\theta_u(t)}{\tilde{\theta}_{\tilde{u}}(t)}$ *is nonincreasing;*

(ii) $\theta_u(t) \le \tilde{\theta}_{\tilde{u}}(t)$ *for* $0 \le t \le e_c(m,u)$;

(iii) $V_m^M(r) \le V_{\tilde{m}}^{\tilde{M}}(r)$ *if* $r \le \min\{e_c(m,u), e_c(\tilde{m},\tilde{u})\}$ *for every unit vector* $u \in M_m$ *and* $\tilde{u} \in \tilde{M}_{\tilde{m}}$.

Proof. This theorem follows from Lemma 8.51 just as Theorem 3.17 follows from Lemma 3.16. ☐

8.10 Problems

8.1 It is possible to estimate the distances $e_f(p,u)$ and $e_c(p,u)$ between P and its focal and cut-focal points in terms of the principal curvatures of P. For $p \in P$ and $u \in P_p^\perp$ let $\kappa_1(u), \dots, \kappa_q(u)$ denote the eigenvalues of T_u. Prove the following theorem (see Hermann [Hr3]):

Assume that M is complete, and let $(p,u) \in \boldsymbol{\nu}$ with $\|u\| = 1$.

(i) *If* $K^M \ge \lambda$, *then*

$$\begin{aligned} e_c(p,u) &\le e_f(p,u) \\ &\le \inf\left\{ \frac{1}{\sqrt{\lambda}} \arctan\left(\frac{\sqrt{\lambda}}{\kappa_a(u)}\right) \;\middle|\; \kappa_a(u) > 0 \text{ for } a = 1,\dots,q \right\}. \end{aligned}$$

(ii) *If* $K^M \le \lambda$, *then*

$$e_f(p,u) \ge \max\left\{ \frac{1}{\sqrt{\lambda}} \arctan\left(\frac{\sqrt{\lambda}}{\kappa_a(u)}\right) \;\middle|\; \kappa_a(u) > 0 \text{ for } a = 1,\dots,q \right\}.$$

(iii) *Suppose that* $\dim P = n-1$, $h > 0$ *and that*

$$\rho^M(x,x) \ge (n-1)\lambda \, \|x\|^2$$

for all unit tangent vectors x to M. Then

$$e_c(p,u) \le e_f(p,u) \le \frac{1}{\sqrt{\lambda}} \arctan\left(\frac{(n-1)\sqrt{\lambda}}{h}\right).$$

8.2 Prove the following theorem of Tsukamoto [Ts], which complements Myers' Theorem (Theorem 3.21, page 49):

Let M be a complete Kähler manifold whose holomorphic sectional curvature is bounded away from zero. Then M is compact.

8.3 The **holomorphic bisectional curvature** of an almost Hermitian manifold is defined to be the sum $K_{xy} + K_{xJy}$ of sectional curvatures, where x, Jx, y, Jy are mutually orthogonal nonzero tangent vectors. Show that if M is a complete Kähler manifold with nonnegative holomorphic bisectional curvature, then for all $r \geq 0$

$$\mathrm{V}_P^M(r) \leq \frac{1}{(n-q)!}\left(\frac{\pi}{\lambda}\sin^2(r\sqrt{\lambda})\right)^{n-q} \mathrm{volume}(P).$$

8.4 Let M be a compact Kähler manifold whose holomorphic and antiholomorphic sectional curvatures satisfy $K_{\mathrm{hol}}^M \geq 4\lambda$ and $K_{\mathrm{ah}}^M \geq \lambda$. Then

$$\mathrm{volume}(M) \leq \mathrm{volume}(\mathbb{C}P^n(\lambda)).$$

(See Theorem 3.22 and [Nay, Theorem 2].)

8.5 Let P be a 2-dimensional compact submanifold of a complete manifold M. Show that $K^M \geq 0$ implies that

$$\begin{aligned}\mathrm{V}_P^M(r) &\leq \frac{(\pi r^2)^{n/2-1}}{\left(\frac{n}{2}-1\right)!}\left\{\mathrm{volume}(P) + \frac{r^2}{n}\left(2\pi\chi(P) - \frac{1}{2}\int_P \tau^M dP\right)\right\}\\ &\leq \frac{(\pi r^2)^{n/2-1}}{\left(\frac{n}{2}-1\right)!}\left\{\mathrm{volume}(P) + \frac{r^2}{4n}\left(\int_P \|H\|^2 dP\right)\right\}\end{aligned}$$

for $0 \leq r \leq \mathrm{minfoc}(P)$. Conclude that

$$\int_P \|H\|^2 dP \geq 2\pi\chi(P) - \frac{1}{2}\int_P \tau^M dP.$$

Chapter 9

Power Series Expansions for Tube Volumes

In this chapter we take a completely different approach to the study of volumes of tubes. We shall compute the first few terms of the power series of the volume function $V_P^M(r)$ as a function of r. In the same issue of the American Journal of Mathematics in which Weyl's paper [Weyl1] appeared in 1939, there is an article [Ht] by the statistician Hotelling.[1] In it Hotelling computes the first two nonzero terms of the expansion for $V_P^M(r)$ in the case that P is a curve in an arbitrary Riemannian manifold M. In fact, Weyl's paper is partially a response to Hotelling's paper. Hotelling discusses several applications of tube formulas to statistics.

But the history of power series expansions of volume functions begins much earlier. In 1848 appeared the article [BDP] by Bertrand,[2] Diguet[3] and Puiseux,[4] in which the first two nonzero terms in the power series expansion for the area of a geodesic ball at a point m in a surface M were computed:

$$\mathrm{V}_m^M(r) = \pi r^2 \left\{1 - \frac{1}{12} K_m r^2 + O(r^4)\right\}. \tag{9.1}$$

(An English translation of [BDP] is given in [Spivak, volume 2, pages 128–131]. See also the discussions in [Monge, pages 583–600] and in [BeGo, page 382].)

[1] Howard Hotelling (1895–1973). American statistician. While majoring in journalism, his analytical abilities were recognized by Eric Temple Bell, who steered him toward mathematics. Although he received his Ph.D. in 1924 with a thesis on topology, he switched to statistics while teaching at Stanford. Hotelling was appointed professor of economics at Columbia in 1931. While there he was able to assist various refugee scholars. His final move was to the University of North Carolina in 1946. He was elected member of the National Academy of Sciences in 1970. Hotelling was a leader in multivariate analysis; in this area his major contribution is called Hotelling's generalized T^2 distribution.

The motivation for proving (9.1) is interesting: to give a new proof that the Gaussian curvature K of a surface in $\mathbb{R}^3$ does not depend on the embedding.[5] In fact, it is obvious from (9.1) that the Gaussian curvature (as well as all the other coefficients in the power series) is intrinsic because $V_m^M(r)$ is.

We shall give a general method for computing the power series expansion of $V_P^M(r)$, where P is a topologically embedded submanifold with compact closure in an arbitrary Riemannian manifold M. But we start out more slowly in Sections 9.1 and 9.2 by doing the case that perhaps has the most interest: the volume of a geodesic ball in a Riemannian manifold. The general case is discussed in Section 9.3.

9.1 Power Series Expansions in Normal Coordinates

Although it is possible to use the formalism of Jacobi fields, normal coordinate vector fields lead more quickly to the computation of the coefficients of a power series in normal coordinates. First, we compute various covariant derivatives of normal coordinate vector fields at the center m of a system of normal coordinates on a Riemannian manifold M. It turns out that these formulas are sufficient to determine completely the power series of any covariant tensor field W in terms of the covariant derivatives of W and the curvature tensor of M at m.

Let M be an arbitrary Riemannian manifold and m a point in M. We define the p^{th} covariant derivative ∇^p inductively as follows. If

$$X_1, \ldots, X_{p+1} \in \mathfrak{X}(M),$$

we put

$$\nabla^p_{X_1 \ldots X_p} X_{p+1} = \nabla_{X_1}\left(\nabla^{p-1}_{X_2 \ldots X_p} X_{p+1}\right).$$

[2] Joseph Louis François Bertrand (1822–1900). French mathematician, Although he began his career as a teacher in secondary schools, he became professor at the École Polytechnique and the Collège de France. In his 1850 paper "Mémoire sur la théorie des courbes à double courboure" Bertrand studied space curves for which there exists a linear relation between the curvature and torsion; such curves are known today as Bertrand curves. In 1856 Bertrand was elected to the Académie des Sciences, and became perpetual secretary in 1874.

In addition to differential geometry, Bertrand worked in number theory and probability theory. His book **Calcul des probabilitiés** was published in 1888. Bertrand wrote may text books and popular books; his lively style always fascinated his readers.

[3] No one seems to know anything else about Diguet, not even his other names.

[4] Victor Alexandre Puiseux (1820–1883). French physicist and mathematician. His scientific work encompassed differential geometry, mechanics, analysis, complex variables, celestial mechanics and observational astronomy. In differential geometry he discovered new properties of evolutes and involutes. Puiseux was a close follower of Cauchy, and in the early 1850's he completed major aspects of theory of functions of a complex variable. His work was surpassed by that of Riemann, at which point Puiseux turned to celestial mechanics.

[5] The pre-Gauss definition of what is now known as the Gaussian curvature of a surface in $\mathbb{R}^3$ was the product of the principal curvatures.

Recall from Chapter 2 (Lemma 2.1, page 15) that the notion of normal coordinate vector field at m does not depend on the choice of normal coordinates centered at m.

In the next few lemmas we compute some of the covariant derivatives of normal coordinate vector fields. This will be useful for computing covariant derivatives of tensor fields such as the curvature tensor and volume form.

Lemma 9.1. *Let X and Y be normal coordinate vector fields at m and ξ be an integral curve of X with $\xi(0) = m$. Then*

$$\left(\nabla^p_{X\ldots X}X\right)_{\xi(t)} = 0 \quad \textit{for } p = 1, 2, \ldots, \tag{9.2}$$

$$\left(\nabla_X Y\right)_m = 0. \tag{9.3}$$

Proof. Since ξ is a geodesic, $\nabla_X X$ vanishes along ξ. Moreover, $(\nabla^p_{X\ldots X}X)_{\xi(t)}$ depends only on the values of X and $\nabla^{p-1}_{X\ldots X}X$ along ξ. From this fact (9.2) follows by induction. Polarization of the equation $(\nabla_X X)_m = 0$ yields

$$\left(\nabla_X Y + \nabla_Y X\right)_m = 0$$

for all normal coordinate vector fields X and Y. We also have

$$\nabla_X Y - \nabla_Y X = [X, Y] = 0, \tag{9.4}$$

because X and Y are constant linear combinations of the coordinate vector fields of a normal coordinate system at m. Hence we get (9.3). $\square$

Next we give formulas for the p^{th} covariant derivative evaluated on $p+1$ normal coordinate vector fields in the case that at most two of them are distinct.

Lemma 9.2. *The following relations among the covariant derivatives of normal coordinate vector fields hold:*

$$\left(\nabla^p_{YX\ldots X}X\right)_m = \cdots = \left(\nabla^p_{X\ldots XYX}X\right)_m, \tag{9.5}$$

$$\left(2\nabla^p_{X\ldots X}Y\right)_m + (p-1)\left(\nabla^p_{X\ldots XYX}X\right)_m = 0, \tag{9.6}$$

$$\left(\nabla^p_{X\ldots X}Y\right)_m = -\left(\frac{p-1}{p+1}\right)\left(\nabla^{p-2}_{X\ldots X}R_{XY}X\right)_m \tag{9.7}$$

for $p \geq 2$.

Proof. Since $[X,Y]=0$, we have

$$R_{XY} = -\nabla_X\nabla_Y + \nabla_Y\nabla_X. \tag{9.8}$$

Let

$$A_k = \left(\nabla^p_{X\ldots Y\ldots X}X\right)_m,$$

where Y occurs in the k^{th} place. From (9.8) it follows that

$$A_k - A_{k-1} = -\left(\nabla^{k-2}_{X\ldots X}R_{XY}\nabla^{p-k}_{X\ldots X}X\right)_m. \tag{9.9}$$

The right-hand side of (9.9) can be expanded in terms of the covariant derivatives of R at m. However, if $2\le k<p$, each term contains a factor of the form

$$\left(\nabla^{p-k}_{X\ldots X}X\right)_m.$$

By Lemma 9.1 each of these factors is zero. Hence $A_k = A_{k-1}$ for $2\le k<p$, and so (9.5) is proved.

Then (9.6) follows from (9.5), the polarized version of (9.2) and (9.3). Finally, (9.7) is an algebraic consequence of (9.5), (9.6) and (9.8). □

It turns out that not all covariant derivatives are needed for the computation of power series in normal coordinates, only those of the form

$$\left(\nabla^p_{X\ldots X}Y\right)_m. \tag{9.10}$$

A general formula for these covariant derivatives is given in [Gr4], but it is too complicated to be useful. We compute the form (9.10) for $p\le 4$.

Lemma 9.3. *If X and Y are normal coordinate vector fields at m, then*

$$\left(\nabla^2_{XY}Z\right)_m = -\frac{1}{3}\left(R_{XY}Z + R_{XZ}Y\right)_m, \tag{9.11}$$

$$\left(\nabla^3_{XXX}Y\right)_m = -\frac{1}{2}\left(\nabla_X(R)_{XY}X\right)_m, \tag{9.12}$$

$$\left(\nabla^4_{XXXX}Y\right)_m = \left(-\frac{3}{5}\nabla^2_{XX}(R)_{XY}X + \frac{1}{5}R_{XR_{XY}X}X\right)_m. \tag{9.13}$$

Proof. From (9.7) follows

$$\left(\nabla^2_{XX}Y\right)_m = -\frac{1}{3}\left(R_{XY}X\right)_m,$$

from which by polarization we obtain

$$\left(\nabla^2_{WX}Y\right)_m + \left(\nabla^2_{XW}Y\right)_m = -\frac{1}{3}\left(R_{WY}X + R_{XY}W\right)_m. \tag{9.14}$$

Also, (9.8) implies that

$$\nabla^2_{WX}Y - \nabla^2_{XW}Y = -R_{WX}Y. \tag{9.15}$$

Then (9.14), (9.15) and the first Bianchi identity (2.19) yield (9.11).

For (9.12) we compute as follows using (9.7):

$$\left(\nabla^3_{XXX}Y\right)_m = -\frac{1}{2}\left(\nabla_X R_{XY}X\right)_m = -\frac{1}{2}\left(\nabla_X(R)_{XY}X\right)_m.$$

Similarly, (9.13) follows from (9.7), but is a little more complicated:

$$\begin{aligned}\left(\nabla^4_{XXXX}Y\right)_m &= -\frac{3}{5}\left(\nabla^2_{XX}R_{XY}X\right)_m\\ &= -\frac{3}{5}\left(\nabla^2_{XX}(R)_{XY}X - \frac{1}{3}R_{X\nabla^2_{XX}Y}X\right)_m\\ &= \left(-\frac{3}{5}\nabla^2_{XX}(R)_{XY}X + \frac{1}{5}R_{XR_{XY}X}X\right)_m. \qquad \square\end{aligned}$$

Assume that M is an analytic Riemannian manifold and that W is an analytic covariant tensor field defined in a neighborhood of m. Let $(x_1, \ldots, x_n)$ be a system of normal coordinates at m and put

$$X_i = \frac{\partial}{\partial x_i}$$

for $i = 1, \ldots, n$. Then $X_1, \ldots, X_n$ are normal coordinate vector fields that are orthonormal at m. Write

$$W(X_{\alpha_1}, \ldots, X_{\alpha_r}) = W_{\alpha_1 \ldots \alpha_r}.$$

Then we have the expansion

$$W_{\alpha_1\ldots\alpha_r} = \sum_{k=0}^{\infty} \sum_{i_1\ldots i_k=1}^{n} \frac{1}{k!}(X_{i_1}\cdots X_{i_k}W_{\alpha_1\ldots\alpha_r})(m)x_{i_1}\cdots x_{i_k}. \tag{9.16}$$

It is a power series on an open subset of $\mathbb{R}^n$ transferred to M by means of normal coordinates. We now express the right-hand side of (9.16) in terms of covariant derivatives evaluated at m.

For any normal coordinate vector field X we have

$$(X^p W_{\alpha_1\ldots\alpha_r})_m = \sum_{\nu_1+\cdots+\nu_{r+1}=p} \frac{p!}{\nu_1!\cdots\nu_{r+1}!}\Big(\nabla^{\nu_{r+1}}_{X\ldots X}(W)(\nabla^{\nu_1}_{X\ldots X}(X_{\alpha_1}), \ldots, \nabla^{\nu_r}_{X\ldots X}(X_{\alpha_r}))\Big)(m). \tag{9.17}$$

Thus $(X^p W_{\alpha_1\ldots\alpha_r})_m$ is computable using Lemmas 9.1–9.3. Next we show how to find the more complicated expressions

$$X_{\beta_1}\ldots X_{\beta_p} W_{\alpha_1\ldots\alpha_r}(m) x_{\beta_1}\cdots x_{\beta_p}$$

using a computational trick.

Lemma 9.4. *Let Φ and Φ' be covariant tensor fields of degree p such that*

$$\Phi(X,\ldots,X)(m) = \Phi'(X,\ldots,X)(m)$$

for all normal coordinate vector fields X at m. Then near m we have

$$\sum_{\beta_1\ldots\beta_p=1}^{n} \Phi(X_{\alpha_1},\ldots,X_{\alpha_p})(m) x_{\beta_1}\cdots x_{\beta_p} = \sum_{\beta_1\ldots\beta_p=1}^{n} \Phi'(X_{\alpha_1},\ldots,X_{\alpha_p})(m) x_{\beta_1}\cdots x_{\beta_p}.$$

Proof. Fix a point m' near to m and put

$$X = \sum_{\alpha=1}^{n} x_\alpha(m') X_\alpha.$$

Because m' is fixed, X is a normal coordinate vector field. Furthermore,

$$\begin{aligned}\sum_{\beta_1\ldots\beta_p=1}^{n} &\Phi(X_{\alpha_1},\ldots,X_{\alpha_p})(m) x_{\beta_1}\cdots x_{\beta_p}(m') \\ &= \Phi(X,\ldots,X)(m) = \Phi'(X,\ldots,X)(m) \\ &= \sum_{\beta_1\ldots\beta_p=1}^{n} \Phi'(X_{\alpha_1},\ldots,X_{\alpha_p})(m) x_{\beta_1}\cdots x_{\beta_p}(m').\end{aligned}$$

Since m' is arbitrary, we get the lemma. ∎

Now we are ready to prove a fundamental theorem about expansions of power series in normal coordinates.

Theorem 9.5. *Let W be a covariant tensor field of degree r, and let $(x_1, \dots, x_n)$ be a system of normal coordinates. Then*

$$W\left(\frac{\partial}{\partial x_{\alpha_1}}, \dots, \frac{\partial}{\partial x_{\alpha_r}}\right)$$

can be expanded in a power series in $x_1, \dots, x_n$ in which the coefficients are expressible in terms of the covariant derivatives of W and R.

Even though it is possible to write down a general expression for the coefficients of the power series (9.16) in terms of the curvature tensor of M at m there is no practical way to find a general formula for all the coefficients. Instead we compute the coefficients in (9.16) up to fourth order. To simplify the notation, let us write

$$\nabla^p_{X_i \dots X_j} = \nabla_{i\dots j}, \qquad R_{X_i X_j X_k X_l} = R_{ijkl},$$

$$\rho(X_i, X_j) = \rho_{ij}, \qquad W(X_{\alpha_1}, \dots, X_{\alpha_r}) = W_{\alpha_1 \dots \alpha_r},$$

and so forth.

Theorem 9.6. *We have the expansion*

$$\begin{aligned} W_{\alpha_1\dots\alpha_r} &= W_{\alpha_1\dots\alpha_r}(m) + \sum_{i=1}^{n} (\nabla_i W_{\alpha_1\dots\alpha_r})(m) x_i \qquad (9.18) \\ &+\frac{1}{2}\sum_{ij=1}^{n}\left\{\nabla_{ij}W_{\alpha_1\dots\alpha_r} - \frac{1}{3}\sum_{a=1}^{r}\sum_{s=1}^{n} R_{i\alpha_a js} W_{\alpha_1\dots\alpha_{a-1}s\alpha_{a+1}\dots\alpha_r}\right\}(m) x_i x_j \\ &+\frac{1}{6}\sum_{ijk=1}^{n}\left\{\nabla_{ijk}W_{\alpha_1\dots\alpha_r} - \sum_{a=1}^{r}\sum_{s=1}^{n} R_{i\alpha_a js}\nabla_k W_{\alpha_1\dots\alpha_{a-1}s\alpha_{a+1}\dots\alpha_r}\right. \\ &\left. -\frac{1}{2}\sum_{a=1}^{r}\sum_{s=1}^{n}\nabla_i R_{j\alpha_a ks} W_{\alpha_1\dots\alpha_{a-1}s\alpha_{a+1}\dots\alpha_r}\right\}(m) x_i x_j x_k \\ &+\text{higher order terms.} \end{aligned}$$

Proof. The constant and linear terms are obvious. To compute the quadratic terms, it suffices by Lemma 9.4 to compute $X_i^2 W_{\alpha_1\dots\alpha_r}(m)$ and linearize it:

$$\begin{aligned} X_i^2 W_{\alpha_1\dots\alpha_r}(m) &= \nabla_{ii} W_{\alpha_1\dots\alpha_r}(m) \qquad (9.19) \\ &\quad + \sum_{s=1}^{n}\sum_{a=1}^{r} \langle \nabla_{ii} X_{\alpha_a}, X_s\rangle (m) W_{\alpha_1\dots\alpha_{a-1}s\alpha_{a+1}\dots\alpha_r}(m) \end{aligned}$$

$$= \left\{ \nabla_{ii} W_{\alpha_1 \ldots \alpha_r} - \frac{1}{3} \sum_{s=1}^{n} \sum_{a=1}^{r} R_{i\alpha_a is} W_{\alpha_1 \ldots \alpha_{a-1} s \alpha_{s+1} \ldots \alpha_r} \right\}(m).$$

Similarly,

$$\begin{aligned} X_i^3 W_{\alpha_1 \ldots \alpha_r}(m) &= \nabla_{iii} W_{\alpha_1 \ldots \alpha_r}(m) \\ &\quad + \sum_{s=1}^{n} \sum_{a=1}^{r} \langle \nabla_{iii} X_{\alpha_a}, X_s \rangle (m) W_{\alpha_1 \ldots \alpha_{a-1} s \alpha_{a+1} \ldots \alpha_r}(m) \\ &\quad + 3 \sum_{s=1}^{n} \sum_{a=1}^{r} \langle \nabla_{ii} X_{\alpha_a}, X_s \rangle (m) \nabla_i W_{\alpha_1 \ldots \alpha_{a-1} s \alpha_{a+1} \ldots \alpha_r}(m) \\ &= \left\{ \nabla_{iii} W_{\alpha_1 \ldots \alpha_r} - \sum_{s=1}^{n} \sum_{a=1}^{r} R_{i\alpha_a is} \nabla_i W_{\alpha_1 \ldots \alpha_{a-1} s \alpha_{a+1} \ldots \alpha_r} \right. \\ &\quad \left. - \frac{1}{2} \sum_{s=1}^{n} \sum_{a=1}^{r} \nabla_i R_{i\alpha_a is} W_{\alpha_1 \ldots \alpha_{a-1} s \alpha_{a+1} \ldots \alpha_r} \right\}(m). \qquad \square \end{aligned}$$

In an important special case the expansion of a tensor field in normal coordinates is much simpler than that of Theorem 9.6.

Corollary 9.7. *In Theorem* 9.6 *assume that* W *is parallel. Then*

$$\begin{aligned} W_{\alpha_1 \ldots \alpha_r} &= W_{\alpha_1 \ldots \alpha_r}(m) \\ &- \frac{1}{6} \sum_{ij=1}^{n} \sum_{a=1}^{r} \sum_{s=1}^{n} R_{i\alpha_a js} W_{\alpha_1 \ldots \alpha_{a-1} s \alpha_{a+1} \ldots \alpha_r}(m) x_i x_j \\ &- \frac{1}{12} \sum_{ijk=1}^{n} \sum_{a=1}^{r} \sum_{s=1}^{n} \nabla_i R_{j\alpha_a ks} W_{\alpha_1 \ldots \alpha_{a-1} s \alpha_{a+1} \ldots \alpha_r}(m) x_i x_j x_k \\ &+ \frac{1}{24} \sum_{ijkl=1}^{n} \left\{ -\frac{3}{5} \sum_{s=1}^{n} \sum_{a=1}^{r} \nabla_{ij} R_{k\alpha_a ls} W_{\alpha_1 \ldots \alpha_{a-1} s \alpha_{a+1} \ldots \alpha_r} \right. \\ &+ \frac{1}{5} \sum_{a=1}^{r} \sum_{st=1}^{n} R_{i\alpha_a js} R_{kslt} W_{\alpha_1 \ldots \alpha_{a-1} t \alpha_{a+1} \ldots \alpha_r} \\ &+ \left. \frac{2}{3} \sum_{1 \le a < b \le r} \sum_{st=1}^{n} R_{i\alpha_a js} R_{k\alpha_b lt} W_{\alpha_1 \ldots \alpha_{a-1} s \alpha_{a+1} \ldots \alpha_{b-1} t \alpha_{b+1} \ldots \alpha_r} \right\}(m) x_i x_j x_k x_l \\ &+ \text{higher order terms.} \end{aligned} \tag{9.20}$$

Proof. It is clear that the linear terms on the right-hand side of (9.18) vanish, and that the quadratic and cubic terms reduce to those on the right-hand side of (9.20). The order 4 terms can be computed by the method of (9.19). $\square$

We give two applications of Corollary 9.7. First, we derive the formula [Ca2, pages 241–244], [Ei, pages 252–256] for the expansion of the metric tensor in normal coordinates. For a system of normal coordinates $(x_1, \ldots, x_n)$ write

$$\left\langle \frac{\partial}{\partial x_p}, \frac{\partial}{\partial x_q} \right\rangle = g_{pq}$$

for $1 \le p, q \le n$.

Corollary 9.8. *The expansion of the component g_{pq} of the metric tensor in normal coordinates about a point m is given by*

$$\begin{aligned} g_{pq} &= \delta_{pq} - \frac{1}{3}\sum_{ij=1}^{n} R_{ipjq}(m)x_ix_j - \frac{1}{6}\sum_{ijk=1}^{n} \nabla_i R_{jpkq}(m)x_ix_jx_k \qquad (9.21) \\ &+ \frac{1}{360}\sum_{ijkl=1}^{n}\left(-18\nabla_{ij}R_{kplq} + 16\sum_{s=1}^{n} R_{ipjs}R_{kqls}\right)(m)x_ix_jx_kx_l \\ &+ \text{higher order terms.} \end{aligned}$$

Proof. Since the metric tensor is parallel, Corollary 9.7 is applicable. Using the facts that $g_{ij}(m) = \delta_{ij}$ and $g_{ij} = g_{ji}$ for $1 \le i, j \le n$, one easily obtains (9.21) as a special case of (9.20). □

The second application of Corollary 9.7 will be a prime ingredient for the computation of the power series for the volume of a small geodesic ball. It is the computation of the power series expansion in normal coordinates of the Riemannian volume element ω of M. Write

$$\omega_{1\ldots n} = \omega(X_1, \ldots, X_n).$$

Corollary 9.9. *The expansion of the volume form ω of a Riemannian manifold M in normal coordinates is*

$$\begin{aligned} \omega_{1\ldots n} &= 1 - \frac{1}{6}\sum_{ij=1}^{n} \rho_{ij}(m)x_ix_j - \frac{1}{12}\sum_{ijk=1}^{n} \nabla_i\rho_{jk}(m)x_ix_jx_k \qquad (9.22) \\ &+ \frac{1}{24}\sum_{ijkl=1}^{n}\left(-\frac{3}{5}\nabla_{ij}\rho_{kl} + \frac{1}{3}\rho_{ij}\rho_{kl} - \frac{2}{15}\sum_{st=1}^{n} R_{isjt}R_{kslt}\right)(m)x_ix_jx_kx_l \\ &+ \text{higher order terms.} \end{aligned}$$

Proof. The Riemannian volume element ω is parallel, and so again we can use Corollary 9.7 to get (9.22). We work out the fourth order term, taking $r = n$ and $W = \omega$ in (9.20). (The second and third order terms are much simpler.)

The coefficient of $x_i x_j x_k x_l$ on the right-hand side of (9.20) has three parts. With $W = \omega$ the first part, when evaluated at m, becomes

$$\left(-\frac{3}{5}\sum_{as=1}^{n} \nabla_{ij} R_{kals}\omega_{1\ldots a-1sa+1\ldots n}\right)(m) = \left(-\frac{3}{5}\sum_{a=1}^{n} \nabla_{ij} R_{kala}\omega_{1\ldots n}\right)(m)$$
$$= \left(-\frac{3}{5}\nabla_{ij}\rho_{kl}\right)(m),$$

and similarly the second part reduces to

$$\left(\frac{1}{5}\sum_{ab=1}^{n} R_{ibja}R_{kalb}\right)(m).$$

The third part is

$$\frac{2}{3}\sum_{1\le a<b\le n}\sum_{st=1}^{n} R_{iajs}R_{kblt}\omega_{1\ldots a-1sa+1\ldots b-1tb+1\ldots n}$$
$$= \frac{2}{3}\sum_{1\le a<b\le n}(R_{iaja}R_{kblb} - R_{iajb}R_{kbla})$$
$$= \frac{1}{3}\left(\rho_{ij}\rho_{kl} - \sum_{ab=1}^{n} R_{iajb}R_{kbla}\right)(m).$$

Since the fourth order term in the expansion of ω is symmetric in x_i, x_j, x_k, x_l, its coefficient is as stated. □

9.2 The Power Series Expansion for the Volume $V_m^M(r)$ of a Small Geodesic Ball

Before proceeding with the calculation of the power series expansion for the volume of a small geodesic ball we need to discuss curvature identities. Most of these have been discussed and used in previous chapters, but we write them down in notation that is convenient for calculational methods of the present chapter.

For $X, Y \in \mathfrak{X}(M)$ the curvature transformation R_{XY} can be extended as a derivation of the whole tensor algebra of M. Let R_{ij} be an abbreviation for $R_{X_iX_j}$.

Lemma 9.10. *We have*

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0 \qquad (\textit{First Bianchi identity}), \tag{9.23}$$

$$\underset{ijk}{\mathfrak{S}}\nabla_i R_{jklp} = 0 \qquad (\textit{Second Bianchi identity}), \tag{9.24}$$

$$\nabla^2_{ij} - \nabla^2_{ji} = -R_{ij} \qquad (\textit{Ricci identity}), \tag{9.25}$$

$$\sum_{i=1}^{n} \nabla_i R_{iakl} = \nabla_k \rho_{al} - \nabla_l \rho_{ak}, \tag{9.26}$$

$$\sum_{i=1}^{n} \nabla_i \rho_{ij} = \frac{1}{2} \nabla_j \tau. \tag{9.27}$$

Proof. For example, we prove (9.26). Using (9.24) and the definition of the Ricci curvature, we compute as follows:

$$\begin{aligned}
\sum_{i=1}^{n} \nabla_i R_{iakl} &= \sum_{i=1}^{n} (-\nabla_l R_{iaik} - \nabla_k R_{iali}) \\
&= \sum_{i=1}^{n} (-\nabla_l R_{iaik} + \nabla_k R_{iail}) \\
&= \nabla_k \rho_{al} - \nabla_l \rho_{ak}.
\end{aligned}$$

Equation (9.27) is proved similarly. □

The first Bianchi identity (9.23) has the following consequences.

Lemma 9.11. *We have*

$$\sum_{abc=1}^{n} R_{abci} R_{acbj} = \frac{1}{2} \sum_{abc=1}^{n} R_{abci} R_{abcj}, \tag{9.28}$$

$$\sum_{abcd=1}^{n} R_{abcd} R_{acbd} = \frac{1}{2} \|R\|^2. \tag{9.29}$$

Proof. Equation (9.28) is a consequence of the following computation:

$$\begin{aligned}
\sum_{abc=1}^{n} R_{abci} R_{acbj} &= \frac{1}{2} \sum_{abc=1}^{n} R_{abci} (R_{acbj} - R_{bcaj}) \\
&= \frac{1}{2} \sum_{abc=1}^{n} R_{abci} (-R_{cabj} - R_{bcaj}) \\
&= \frac{1}{2} \sum_{abc=1}^{n} R_{abci} R_{abcj}.
\end{aligned}$$

Then (9.29) results when (9.28) is contracted. □

Now we are ready to find the power series expansion for $V_m^M(r)$.

Theorem 9.12. *Let M be an n-dimensional Riemannian manifold, and let $m \in M$. Then for sufficiently small r we have*

$$\mathrm{V}_m^M(r) = \frac{(\pi r^2)^{\frac{n}{2}}}{(\frac{1}{2}n)!}\Bigg\{1 - \frac{\tau\, r^2}{6(n+2)} \tag{9.30}$$

$$+\frac{\Big(-3\|R\|^2 + 8\|\rho\|^2 + 5\tau^2 - 18\Delta\tau\Big)r^4}{360(n+2)(n+4)} + O(r^6)\Bigg\}_m.$$

Proof. Let $(x_1, \ldots, x_n)$ be any normal coordinate system at m, and write $x_i = a_i r$ for $i = 1, \ldots, n$. From Corollary 9.9 we obtain the expansion

$$\omega_{1\ldots n}(\exp_m(ru)) = \sum_{p=0}^{\infty} \frac{\mu_p}{p!} r^p, \tag{9.31}$$

where $\mu_0 = 1$, $\mu_1 = 0$,

$$\mu_2 = -\frac{1}{3}\sum_{ij=1}^{n} \rho_{ij} a_i a_j, \qquad \mu_3 = -\frac{1}{2}\sum_{ijk=1}^{n} \nabla_i \rho_{jk} a_i a_j a_k,$$

and

$$\mu_4 = \sum_{ijkl=1}^{n}\Bigg\{-\frac{3}{5}\nabla_{ij}\rho_{kl} + \frac{1}{3}\rho_{ij}\rho_{kl} - \frac{2}{15}\sum_{st=1}^{n} R_{isjt}R_{kslt}\Bigg\} a_i a_j a_k a_l. \tag{9.32}$$

(We have simplified the notation by making the convention that all coefficients are evaluated at m.)

The power series expansion (9.31) and formula (3.20) on page 41 yield

$$\begin{aligned} \mathrm{A}_m^M(r) &= r^{n-1}\int_{S^{n-1}(1)} \omega_{1\ldots n}(\exp_m(ru))\,du \\ &= r^{n-1}\sum_{p=0}^{\infty}\frac{r^p}{p!}\int_{S^{n-1}(1)} \mu_p\, du. \end{aligned} \tag{9.33}$$

The integrals of the μ_i can be computed from the moment formulas from Section A.2 of the Appendix. First of all, the integral of each of the μ_{2p+1}'s is zero, because the integral over one hemisphere cancels the integral over the other. Furthermore,

$$\int_{S^{n-1}(1)} \mu_0\, du = \text{volume}(S^{n-1}(1)) = \frac{2\,\pi^{n/2}}{(\frac{1}{2}(n-1))!}. \tag{9.34}$$

Next

$$\int_{S^{n-1}(1)} \mu_2\,du = -\frac{1}{3}\sum_{ij=1}^{n}\int_{S^{n-1}(1)} \rho_{ij}a_ia_j\,du \tag{9.35}$$

$$= -\frac{1}{3}\sum_{i=1}^{n}\rho_{ii}\int_{S^{n-1}(1)} a_i^2\,du$$

$$= \frac{-\pi^{n/2}\,\tau}{3(\frac{1}{2}n)!},$$

because $a_1^2+\cdots+a_n^2=1$. For μ_4 we note that from equations (A.4) and (A.5) of the Appendix it follows that

$$\int_{S^{n-1}(1)} a_i^4\,du = 3\int_{S^{n-1}(1)} a_i^2a_j^2\,du = \frac{3\,\pi^{n/2}}{(n+2)(\frac{1}{2}n)!} \tag{9.36}$$

for $i\neq j$. Write

$$\lambda_{ijkl} = -\frac{3}{5}\nabla_{ij}\rho_{kl}+\frac{1}{3}\rho_{ij}\rho_{kl}-\frac{2}{15}\sum_{st=1}^{n}R_{isjt}R_{kslt}.$$

Then making extensive use of (9.23)–(9.29) and (9.36), we compute

$$\int_{S^{n-1}(1)} \mu_4\,du = \sum_{ijkl=1}^{n}\lambda_{ijkl}\int_{S^{n-1}(1)} a_ia_ja_ka_l\,du \tag{9.37}$$

$$= \frac{\pi^{n/2}}{(n+2)(\frac{1}{2}n)!}\left\{3\sum_{i=1}^{n}\lambda_{iiii}+\sum_{i\neq j}(\lambda_{iijj}+\lambda_{ijij}+\lambda_{ijji})\right\}$$

$$= \frac{\pi^{n/2}}{(n+2)(\frac{1}{2}n)!}\sum_{ij=1}^{n}(\lambda_{iijj}+\lambda_{ijij}+\lambda_{ijji})$$

$$= \frac{\pi^{n/2}}{(n+2)(\frac{1}{2}n)!}\sum_{ij=1}^{n}\left\{-\frac{3}{5}\nabla_{ii}\rho_{jj}-\frac{6}{5}\nabla_{ij}\rho_{ij}+\frac{1}{3}\rho_{ii}\rho_{jj}\right.$$

$$\left.+\frac{2}{3}\rho_{ij}^2-\frac{2}{15}\sum_{st=1}^{n}(R_{isit}R_{jsjt}+R_{isjt}^2+R_{isjt}R_{itjs})\right\}$$

$$= \frac{\pi^{n/2}}{15(n+2)(\frac{1}{2}n)!}\left\{5\tau^2+8\|\rho\|^2-3\|R\|^2-18\Delta\tau\right\}.$$

Finally, from (9.34)–(9.37) and (3.23) we obtain (9.30). □

The generalization of (9.1) to general Riemannian manifolds (that is, the first two terms of (9.30)) was first given in a little known paper by H. Vermeil [Ve1], which Cartan mentions [Ca2, page 256], but Hotelling does not. The quadratic term (9.30) is given in [Gr4] and one additional term is given in [GV1]. Four terms in the expansion for surfaces are given in [GV3].

The expansion (9.30) can be used to obtain a local comparison theorem.

Corollary 9.13. *Let M be an analytic Riemannian manifold and let $m \in M$.*

(i) $\tau_m > 0$ *if and only if*

$$V_m^M(r) < \frac{(\pi r^2)^{n/2}}{(\frac{1}{2}n)!}$$

for all sufficiently small $r > 0$.

(ii) $\tau_m < 0$ *if and only if*

$$V_m^M(r) > \frac{(\pi r^2)^{n/2}}{(\frac{1}{2}n)!}$$

for all sufficiently small $r > 0$.

It is interesting that Corollary 9.13 is neither stronger nor weaker than the Bishop-Günther inequalities (Theorems 3.17 and 3.19). On the one hand, Corollary 9.13 holds only for sufficiently small $r > 0$, while the Bishop-Günther inequalities are valid for r up to the first conjugate point. On the other hand, the condition that the scalar curvature τ be positive at m is weaker than positivity at m of either the sectional curvature or Ricci curvature. (See problem 3.12.)

The following conjecture is true in dimensions 2 and 3, but is unresolved in higher dimensions:

Conjecture. *Suppose that $V_m^M(r)$ coincides with the volume of a geodesic ball in Euclidean space for all sufficiently small r and all $m \in M$, that is,*

$$V_m^M(r) = \frac{(\pi r^2)^{n/2}}{(\frac{1}{2}n)!}$$

for all sufficiently small $r > 0$ and all $m \in M$. Then M is flat.

The conjecture holds for 2-dimensional manifolds because the scalar curvature determines the sectional curvature (in fact, the scalar curvature is just twice the sectional curvature). Moreover, for 2-dimensional manifolds the conjecture follows from (9.1). Notice that only the first two terms in the expansion of $V_m^M(r)$ are needed.

The proof of the conjecture for 3-dimensional manifolds is a little more complicated. It depends on the fact that for a 3-dimensional Riemannian manifold the curvature tensor is completely determined by the Ricci curvature. The exact formula is given in the following lemma:

Lemma 9.14. *The curvature tensor of a 3-dimensional Riemannian manifold M satisfies*

$$R_{abcd} = \rho_{ac}\delta_{bd} + \rho_{bd}\delta_{ac} - \rho_{ad}\delta_{bc} - \rho_{bc}\delta_{ad} - \frac{\tau}{2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}), \tag{9.38}$$

$$\|R\|^2 = 4\|\rho\|^2 - \tau^2. \tag{9.39}$$

Proof. The easiest way to prove (9.38) is to make use of the fact that the Weyl curvature tensor vanishes identically for a 3-dimensional Riemannian manifold. Then (9.39) can be proved directly from (9.38); however, it is simpler to deduce it from the fact that the 4-dimensional Gauss-Bonnet integrand vanishes identically on any 3-dimensional manifold. □

It follows from (9.39) and (9.30) that when $\dim M = 3$ the power series expansion for $V_m^M(r)$ is expressible entirely in terms of the Ricci and scalar curvatures. In fact,

Corollary 9.15. *Let M be a 3-dimensional Riemannian manifold, and let $m \in M$. Then*

$$V_m^M(r) = \frac{4\pi r^3}{3}\left\{1 - \frac{\tau}{30}r^2 + \frac{1}{6300}\left(4\tau^2 - 2\|\rho\|^2 - 9\Delta\tau\right)r^4 + O(r^6)\right\}_m. \tag{9.40}$$

Corollary 9.16. *Suppose* $\dim M = 3$ *and $V_m^M(r)$ identically equals $(4\pi/3)r^3$. Then M is flat.*

Proof. It follows from (9.40) that τ and $\|\rho\|^2$ vanish identically. But then (9.38) implies that $\|R\|^2$ also vanishes identically. Thus M is flat. □

In [GV3] is given an example of a nonflat homogeneous manifold M for which

$$V_m^M(r) = \frac{(\pi r^2)^{\frac{n}{2}}}{(\frac{1}{2}n)!}\left\{1 + O(r^8)\right\}$$

at all points m. O. Kowalski [Kw] has found manifolds for which the volume of a geodesic ball at each point approximates the volume of a geodesic ball in $\mathbb{R}^n$ to an even higher degree. For the proof of the conjecture in some other special cases, see [CaVa] and [FV].

9.3 Power Series Expansions in Fermi Coordinates

The method of the previous section needs only slight modification in order to work for Fermi coordinates. We shall see that versions of Lemmas 9.1–9.3 hold for normal Fermi fields. In addition, certain covariant derivatives containing both tangential and normal Fermi fields are needed, but these are not difficult to compute.

In what follows we use the notation of Section 2.2 of Chapter 2. In particular, we denote by $\mathfrak{X}(P,m)^\perp$ and $\mathfrak{X}(P,m)^\top$ the spaces of normal and tangential Fermi fields at $m \in P$.

Lemma 9.17. *Let $X \in \mathfrak{X}(P,m)^\perp$ and $A \in \mathfrak{X}(P,m)^\top$.*

(i) *If $\xi\colon (a,b) \longrightarrow M$ is a geodesic normal to P such that $\xi(0) \in P$ and $X_{\xi(t)} = \xi'(t)$ for $a < t < b$, then*

$$\left(\nabla^p_{X\dots X} X\right)_{\xi(t)} = 0. \tag{9.41}$$

(ii) *If $\eta\colon (c,d) \longrightarrow P$ is a curve such that $\eta(0) = p$ and $\eta'(t) = A_{\eta(t)}$, then*

$$\left(\nabla^p_{A\dots AX\dots X} X\right)_{\eta(t)} = 0, \tag{9.42}$$

provided that at least two X's appear on the left-hand side of (9.42).

Proof. The proof of (9.41) is the same as that of (9.2). Then (9.42) is obtained by taking successive covariant derivatives of

$$\left(\nabla_X X\right)_{\eta(t)} = 0$$

with respect to A. □

Lemma 9.18. *Let X, Y, A and $\eta(t)$ be as in Lemma* 9.17, *and let $p \geq 2$. Then*

$$\begin{aligned}\left(\nabla^p_{YX\dots X} X\right)_{\eta(t)} &= \left(\nabla^p_{XYX\dots X} X\right)_{\eta(t)} \\ &= \dots = \left(\nabla^p_{X\dots YX} X\right)_{\eta(t)};\end{aligned} \tag{9.43}$$

$$\left(\nabla^p_{X\dots XY} X\right)_{\eta(t)} = \left(\nabla^p_{X\dots X} Y\right)_{\eta(t)}; \tag{9.44}$$

$$\left(\nabla^p_{AX\dots X} X\right)_{\eta(t)} = \dots = \left(\nabla^p_{X\dots XAX} X\right)_{\eta(t)} = 0; \tag{9.45}$$

$$\left(\nabla^p_{X\dots XA} X\right)_{\eta(t)} = \left(\nabla^p_{X\dots X} A\right)_{\eta(t)}. \tag{9.46}$$

Proof. Equations (9.44) and (9.46) are immediate consequences of (9.4); the proof of (9.43) is the same as that of (9.5). To prove (9.45), we put

$$B_s = \left(\nabla^p_{X\ldots A\ldots X} X\right)_{\eta(t)},$$

where A occurs in the s^{th} place. Since $[X, A] = 0$ and (9.8) holds for normal Fermi fields, it follows that

$$B_s - B_{s-1} = -\left(\nabla^{s-2}_{X\ldots X}\left(R_{XA}\left(\nabla^{p-s}_{X\ldots X} X\right)\right)\right)_{\eta(t)}. \tag{9.47}$$

The right-hand side of (9.47) can be expressed in terms of the covariant derivatives of R, A and X at $\eta(t)$. Just as in the proof of (9.5), each of these terms contains a factor of the form $\nabla^q_{X\ldots X}X$; here $q > 0$, provided $2 \le s < p$. On the other hand, $B_1 = 0$ by (9.42). Hence we get (9.45). □

Lemma 9.19. *Let X, Y, A be as in Lemma* 9.17. *Then*

$$\left(\nabla_X Y\right)_m = 0; \tag{9.48}$$

$$\left(\nabla^p_{X\ldots X} Y\right)_m = -\left(\frac{p-1}{p+1}\right)\left(\nabla^{p-2}_{X\ldots X} R_{XY} X\right)_m; \tag{9.49}$$

$$\left(\nabla^p_{X\ldots X} A\right)_m = -\left(\nabla^{p-2}_{X\ldots X} R_{XA} X\right)_m \quad \textit{for } p \ge 2. \tag{9.50}$$

Proof. The proofs of (9.48) and (9.49) are the same as those of (9.3) and (9.7). For (9.50), we use (9.45) and (9.46) to obtain

$$\begin{aligned}
\left(\nabla^p_{X\ldots X} A\right)_m &= \left(\nabla^p_{X\ldots XA} X\right)_m \\
&= -\left(\nabla^{p-2}_{X\ldots X} R_{XA} X\right)_m + \left(\nabla^p_{X\ldots AX} X\right)_m \\
&= -\left(\nabla^{p-2}_{X\ldots X} R_{XA} X\right)_m.
\end{aligned}$$

□

Next we write down explicit formulas for some of the covariant derivatives of order less than or equal to two.

Lemma 9.20. *Let $A \in \mathfrak{X}(P,m)^{\top}$ and $X, Y, Z \in \mathfrak{X}(P,m)^{\perp}$. Then*

$$\left(\nabla^2_{XY} A\right)_m = -\left(R_{XA} Y\right)_m; \tag{9.51}$$

$$\left(\nabla^2_{XY} Z\right)_m = -\frac{1}{3}\left(R_{XY} Z + R_{XZ} Y\right)_m. \tag{9.52}$$

Proof. The proof of (9.52) is the same as that of (9.11), and (9.51) follows from (9.50). $\square$

The result corresponding to Theorem 9.5 is:

Theorem 9.21. *Let W be a covariant tensor field of degree r and suppose $(x_1, \ldots, x_n)$ is a system of Fermi coordinates. Then*

$$W\left(\frac{\partial}{\partial x_{\alpha_1}}, \ldots, \frac{\partial}{\partial x_{\alpha_r}}\right)$$

can be expanded in a power series in $x_1, \ldots, x_n$ in which the coefficients are expressible in terms of the covariant derivatives of W, T and R.

Getting the general coefficient in a power series expansion in Fermi coordinates is even more impractical than for normal coordinates. Since we are interested in computing volumes of tubes, we limit ourselves to finding the first three terms in the expansion of the volume form. Recall that H denotes the mean curvature vector field.

Theorem 9.22. *Let P be a submanifold of M and $m \in P$. Then the power series of $\omega_{1\ldots n}$ in the Fermi coordinates $x_{q+1}, \ldots, x_n$ is given by*

$$\begin{aligned}\omega_{1\ldots n} &= 1 - \sum_{i=q+1}^{n} \langle H, X_i\rangle (m) x_i \qquad (9.53)\\ &- \frac{1}{6}\sum_{ij=q+1}^{n}\left\{\rho_{ij} + 2\sum_{a=1}^{q} R_{aiaj} - 3\sum_{ab=1}^{q}\Big(T_{aai}T_{bbj} - T_{abi}T_{abj}\Big)\right\}(m)x_i x_j\\ &+ \text{ higher order terms.}\end{aligned}$$

Proof. It is clear that $\omega_{1\ldots n}(m) = 1$ because the Fermi fields X_a are orthonormal at m. Moreover, let $X \in \mathfrak{X}(P, m)^{\perp}$; from (9.48) we have

$$\begin{aligned}(X\omega_{1\ldots n})(m) &= \sum_{\alpha=1}^{n} \omega\Big(X_1, \ldots, \nabla_X X_\alpha, \ldots, X_n\Big)(m)\\ &= \sum_{\alpha=1}^{n}\Big\langle \nabla_X X_\alpha, X_\alpha\Big\rangle(m)\\ &= \sum_{a=1}^{q}\Big\langle \nabla_X X_a, X_a\Big\rangle(m).\end{aligned}$$

Also, $\left\langle \nabla_X A, A\right\rangle = \left\langle \nabla_A X, A\right\rangle = -\left\langle T_A A, X\right\rangle$ for $A \in \mathfrak{X}(P,m)^\top$. So

$$(X\omega_{1\dots n})(m) = -\sum_{a=1}^{q} T_{aaX}(m) = -\langle H, X\rangle (m).$$

Similarly, using (9.51) and (9.52) we find that

$$\begin{aligned}(X^2\omega_{1\dots n}) &= \Bigg\{ \sum_{\alpha=1}^{n} \left\langle \nabla^2_{XX} X_\alpha, X_\alpha \right\rangle \\ &\quad + \sum_{\alpha\beta=1}^{n} \det \begin{pmatrix} \left\langle \nabla_X X_\alpha, X_\alpha\right\rangle & \left\langle \nabla_X X_\alpha, X_\beta\right\rangle \\ \left\langle \nabla_X X_\beta, X_\alpha\right\rangle & \left\langle \nabla_X X_\beta, X_\beta\right\rangle \end{pmatrix} \Bigg\}(m). \\ &= \left\{ -\sum_{a=1}^{q} R_{XaXa} - \frac{1}{3}\sum_{i=q+1}^{n} R_{XiXi} + \sum_{ab=1}^{q}\left(T_{aaX}T_{bbX} - T^2_{abX}\right)\right\}(m) \\ &= \left\{ -\frac{1}{3}\rho(X,X) - \frac{2}{3}\sum_{a=1}^{q} R_{XaXa} + \sum_{ab=1}^{q}\left(T_{aaX}T_{bbX} - T^2_{abX}\right)\right\}(m).\end{aligned}$$

Hence using Theorem 9.21, we get (9.53). □

We are now ready to derive the power series expansion in r of the volume of a tube about a submanifold P of a Riemannian manifold M.

Theorem 9.23. *We have*

$$V_P^M(r) = \frac{(\pi r^2)^{\frac{1}{2}(n-q)}}{(\frac{1}{2}(n-q))!}\int_P \left\{1 + Ar^2 + Br^4 + O(r^6)\right\} dP, \tag{9.54}$$

where[6]

$$A = \frac{1}{2(n-q+2)}\left(\tau(R^P) - \sum_{ab=1}^{q} R^M_{abab} - \sum_{a=1}^{q}\sum_{i=q+1}^{n} R^M_{aiai} - \frac{1}{3}\sum_{ij=q+1}^{n} R^M_{ijij}\right)_m. \tag{9.55}$$

Proof. Just as in the proof of Theorem 9.12, we write

$$\omega_{1\dots n}(\exp_m(ru)) = \sum_{p=0}^{\infty} \frac{\mu_p}{p!} r^p,$$

[6]The formula for the coefficient B of r^4 in (9.54) is a very complicated expression involving R^P, R^M and also the second fundamental form of P in M. Although A is independent of the second fundamental form, B may not be.

but this time we use Fermi coordinates instead of normal coordinates. Then from (9.53) we have

$$\mu_0 = 1, \qquad \mu_1 = -\sum_{i=q+1}^{n} \langle H, X_i\rangle(m)a_i,$$

and

$$\begin{aligned} 3\mu_2 = -\sum_{ij=q+1}^{n} \Bigg\{\rho_{ij}^M + 2\sum_{a=1}^{q} R_{iaja}^M \\ -3\sum_{ab=1}^{q}(T_{aai}T_{bbj} - T_{abi}T_{abj})\Bigg\}(m)a_ia_j. \end{aligned} \tag{9.56}$$

It follows from (3.20) that

$$\begin{aligned} \mathrm{A}_P^M(r) &= r^{n-q-1}\int_{S^{n-q-1}(1)} \omega_{1\dots n}(\exp_m(ru))\,du \\ &= r^{n-q-1}\sum_{p=0}^{\infty}\frac{r^p}{p!}\int_{S^{n-q-1}(1)} \mu_p\,du. \end{aligned} \tag{9.57}$$

Just as in the proof of Theorem 9.12, the integrals of all the μ_p are zero for p odd. Furthermore,

$$\int_{S^{n-q-1}(1)} \mu_0\,du = \text{volume}(S^{n-q-1}(1)) = \frac{2\,\pi^{\frac{1}{2}(n-q)}}{\Gamma(\frac{1}{2}(n-q))}. \tag{9.58}$$

To find the integral of μ_2, we use (9.56), the Gauss equation (4.28), and compute as follows:

$$\begin{aligned} \int_{S^{n-q-1}(1)} \mu_2\,du &= -\frac{1}{3}\sum_{ij=q+1}^{n}\int_{S^{n-q-1}(1)} \Bigg\{\rho_{ij}^M + 2\sum_{a=1}^{q} R_{iaja}^M \\ &\qquad -3\sum_{ab=1}^{q}(T_{aai}T_{bbj} - T_{abi}T_{abj})\Bigg\}(m)a_ia_j\,du \\ &= -\frac{1}{3}\sum_{i=q+1}^{n}\Bigg\{\rho_{ii}^M + 2\sum_{a=1}^{q} R_{iaia}^M - 3\sum_{ab=1}^{q}(T_{aai}T_{bbi} - T_{abi}^2)\Bigg\}\int_{S^{n-q-1}(1)} a_i^2\,du \\ &= \frac{-\pi^{\frac{1}{2}(n-q)}}{3(\frac{1}{2}(n-q))!}\Bigg\{\tau(R^M) + \sum_{a=1}^{q}\rho_{aa}^M + \sum_{ab=1}^{q} R_{abab}^M - 3\tau(R^P)\Bigg\}. \end{aligned}$$

Thus we obtain (9.54). □

The power series for $V_P^M(r)$ when P is 1-dimensional is especially simple (see [Ht]):

Corollary 9.24. (Hotelling's Tube Formula.) *Let ξ be a unit-speed curve of finite length $L(\xi)$ in a Riemannian manifold M. Then the volume of a tube of radius r about ξ is given by*

$$\begin{aligned} V_\xi^M(r) &= \frac{(\pi r^2)^{\frac{1}{2}(n-1)}}{(\frac{1}{2}(n-1))!}\Big\{L(\xi) \\ &\quad -\frac{r^2}{6(n+1)}\int_\xi \big(\tau(R^M)\circ\xi+\rho^M(\xi',\xi')\big)\,dt+O(r^4)\Big\}. \end{aligned} \tag{9.59}$$

Proof. When $\dim P = 1$, we have $\tau(R^P) = R_{abab}^M = 0$. Therefore, the coefficient A in (9.55) reduces to

$$\begin{aligned} A &= \frac{1}{2(n+1)}\left(-\sum_{i=2}^{n} R_{1i1i}^M - \frac{1}{3}\sum_{ij=2}^{n} R_{ijij}^M\right) \\ &= \frac{1}{2(n+1)}\left(-\frac{1}{3}\rho^M(\xi',\xi') - \frac{1}{3}\sum_{ij=1}^{n} R_{ijij}^M\right) \\ &= -\frac{1}{6(n+1)}\left(\rho^M(\xi',\xi')+\tau(R^M)\circ\xi\right), \end{aligned}$$

and so we get (9.59). □

Finally, we get an estimate for the volume of a tube of small radius in terms of sectional curvature. In contrast to Corollary 8.6 there are no dimension restrictions.

Corollary 9.25. *Let P be a topologically embedded submanifold of a Riemannian manifold M. Then:*

(i) $K^M > 0$ *implies that* $V_P^M(r) < V_P^{\mathbb{R}^n}(r)$;

(ii) $K^M < 0$ *implies that* $V_P^M(r) > V_P^{\mathbb{R}^n}(r)$.

9.4 Problems

9.1 Show that the third term in the expansion for $\omega_{1\ldots n}$ in Fermi coordinates is

$$-\frac{1}{12}\sum_{ijk=q+1}^{n}\Bigg\{\nabla_i\rho_{jk}-2\rho_{ij}\langle H,k\rangle+\sum_{a=1}^{q}\Big(\nabla_i R_{ajak}-4R_{iaja}\langle H,k\rangle\Big)$$

$$+4\sum_{ab=1}^{q}R_{iajb}T_{abk}+2\sum_{abc=1}^{q}\Big(T_{aai}T_{bbj}T_{cck}$$

$$-3T_{aai}T_{bcj}T_{bck}+2T_{abi}T_{bcj}T_{cak}\Big)\Bigg\}(m)x_ix_jx_k.$$

9.2 Show that the volume of a tube about a curve ξ in a Kähler manifold $\mathbb{K}^n_{\mathrm{hol}}(\lambda)$ of constant holomorphic sectional curvature 4λ is given by

$$\frac{d}{dr}V_\xi^{\mathbb{K}^n_{\mathrm{hol}}(\lambda)}(r)=\frac{2\pi^{n-\frac{1}{2}}}{(n-\frac{3}{2})!\lambda^{n-1}}\sin^{2n-2}(\sqrt{\lambda}r)$$

$$\cdot\left\{1-\frac{2n}{2n-1}\sin^2(\sqrt{\lambda}r)\right\}L(\xi).$$

9.3 Let $\mathbb{Q}P^n(\lambda)$ denote quaternionic projective space with the natural metric with maximum sectional curvature 4λ. (It has real dimension $4n$.)

a. Show that the volume of a geodesic ball of radius r about any point $m\in\mathbb{Q}P^n(\lambda)$ is given by

$$V_m^{\mathbb{Q}P^n(\lambda)}(r)=\frac{\pi^{2n}}{(2n+1)!\lambda^{2n}}\sin^{4n}(\sqrt{\lambda}r)\left\{1+2n\cos^2(\sqrt{\lambda}r)\right\}.$$

b. Show that the volume of a tube of radius r about a curve ξ in $\mathbb{Q}P^n(\lambda)$ is given by

$$\frac{d}{dr}V_\xi^{\mathbb{Q}P^n(\lambda)}(r)=\frac{2\pi^{2n-\frac{1}{2}}}{(2n-\frac{3}{2})!\lambda^{2n-1}}\sin^{4n-2}(\sqrt{\lambda}r)$$

$$\cdot\left\{1-\frac{8n+1}{4n-1}\sin^2(\sqrt{\lambda}r)+\frac{4n+2}{4n-1}\sin^4(\sqrt{\lambda}r)\right\}L(\xi).$$

c. Show that the volume of quaternionic projective space $\mathbb{Q}P^n(\lambda)$ is given by

$$\mathrm{volume}(\mathbb{Q}P^n(\lambda))=V_m^{\mathbb{Q}P^n(\lambda)}\left(\frac{\pi}{2\sqrt{\lambda}}\right)=\frac{1}{(2n+1)!}\left(\frac{\pi}{\lambda}\right)^{2n}.$$

9.4 Let $\mathbb{C}\mathrm{ay}P^2(\lambda)$ denote the Cayley projective plane with natural metric with maximum sectional curvature 4λ. (It has real dimension 16.)

a. Show that the volume of a geodesic ball of radius r at any point $m \in \mathbb{C}\mathrm{ay}P^2(\lambda)$ is given by

$$V_m^{\mathbb{C}\mathrm{ay}P^2(\lambda)}(r) = \frac{6\pi^8}{11!\lambda^8}\sin^{16}(\sqrt{\lambda}r)\Big\{1 + 8\cos^2(\sqrt{\lambda}r) + 36\cos^4(\sqrt{\lambda}r) + 120\cos^6(\sqrt{\lambda}r)\Big\}.$$

b. Show that the volume of a tube of radius r about a curve ξ in $\mathbb{C}\mathrm{ay}P^2(\lambda)$ is given by

$$\frac{d}{dr}V_\xi^{\mathbb{C}\mathrm{ay}P^2(\lambda)}(r) = \frac{2\pi^{\frac{15}{2}}}{(\frac{13}{2})!\lambda^7}\sin^{14}(\sqrt{\lambda}r)\Big\{1 - \frac{67}{15}\sin^2(\sqrt{\lambda}r) + \frac{37}{5}\sin^4(\sqrt{\lambda}r) - \frac{27}{5}\sin^6(\sqrt{\lambda}r) + \frac{22}{15}\sin^8(\sqrt{\lambda}r)\Big\}L(\xi).$$

c. Show that the volume of the Cayley projective plane $\mathbb{C}\mathrm{ay}P^2(\lambda)$ is given by

$$\mathrm{volume}(\mathbb{C}\mathrm{ay}P^2(\lambda)) = V_m^{\mathbb{C}\mathrm{ay}P^2(\lambda)}\left(\frac{\pi}{2\sqrt{\lambda}}\right) = \frac{6}{11!}\left(\frac{\pi}{\lambda}\right)^8.$$

9.5 Show that the fifth order term in the expansion (9.21) for g_{pq} is given by

$$\frac{1}{90}\sum_{ijklh=1}^{n}\Big\{-\nabla_{ijk}R_{lphq} + 2\sum_{s=1}^{n}\Big(\nabla_i R_{jpks}R_{lqhs} + \nabla_i R_{jqks}R_{lphs}\Big)\Big\}(m)x_ix_jx_kx_lx_h.$$

9.6 Show that the fifth and sixth order terms in the expansion (9.22) for $\omega_{1\ldots n}$ are given by

$$\frac{1}{360}\sum_{ijklh=1}^{n}\Big\{-2\nabla_{ijk}\rho_{lh} + 5\nabla_i\rho_{jk}\rho_{lh} - 2\sum_{st=1}^{n}\nabla_i R_{jskt}R_{lsht}\Big\}(m)x_ix_jx_kx_lx_h$$

and

$$\frac{1}{720}\sum_{ijklhg=1}^{n}\left\{-\frac{5}{7}\nabla_{ijkl}\rho_{hg}+3(\nabla_{ij}\rho_{kl})\rho_{hg}+\frac{5}{2}(\nabla_i\rho_{jk})(\nabla_l\rho_{hg})\right.$$

$$-\frac{8}{7}\sum_{ab=1}^{n}(\nabla_{ij}R_{kalb})R_{hagb}-\frac{5}{9}\rho_{ij}\rho_{kl}\rho_{hg}-\frac{15}{14}\sum_{ab=1}^{n}(\nabla_i R_{jakb})(\nabla_l R_{hagb})$$

$$\left.-\frac{16}{63}\sum_{abc=1}^{n}R_{iajb}R_{kblc}R_{hcga}+\frac{2}{3}\rho_{ij}\sum_{ab=1}^{n}R_{kalb}R_{hagb}\right\}(m)x_i x_j x_k x_l x_h x_g.$$

9.7 Using the notation of problem 4.8, show that the coefficient of the fourth term in the expansion (9.30) for the volume of a small geodesic ball is

$$\frac{1}{720(n+2)(n+4)(n+6)}\left\{-\frac{5}{9}\tau^3-\frac{8}{3}\tau\|\rho\|^2+\tau\|R\|^2+\frac{64}{63}\check{\rho}\right.$$

$$-\frac{64}{21}\langle\rho\otimes\rho,\bar{R}\rangle+\frac{32}{7}\left\langle\rho,\dot{R}\right\rangle-\frac{110}{63}\check{R}-\frac{200}{63}\check{\bar{R}}+\frac{45}{7}\|\nabla\tau\|^2+\frac{45}{14}\|\nabla\rho\|^2$$

$$+\frac{45}{7}\alpha(\rho)-\frac{45}{14}\|\nabla R\|^2+6\tau\Delta\tau+\frac{48}{7}\langle\Delta\rho,\rho\rangle+\frac{54}{7}\left\langle\nabla^2\tau,\rho\right\rangle$$

$$\left.-\frac{30}{7}\langle\Delta R,R\rangle-\frac{45}{7}\Delta^2\tau\right\}(m).$$

Chapter 10

Steiner's Formula

In 1840 J. Steiner [Sr] studied convex regions in 2- and 3-dimensional Euclidean space; he obtained a formula for the volume of the convex region B_r consisting of those points whose distance from a given convex region B is less than or equal to r. In this chapter we put Steiner's Formula into the same general framework as Weyl's Tube Formula.

Steiner[1] was actually concerned with convex regions whose boundaries are polygons. Let us define a **region** B in a manifold M to be an open subset of M with compact closure. Denote by ∂B the boundary of B. A region in $\mathbb{R}^n$ is called **convex** provided any two points in it can be joined by a straight line lying entirely within the region. Using the methods of Euclidean geometry, Steiner proved the following two results:

Theorem 10.1. *Let B be a convex region in the plane. Then for all $r \geq 0$*

$$\operatorname{Area}(B_r) = \operatorname{Area}(B) + \operatorname{Length}(\partial B)r + \pi r^2; \tag{10.1}$$

$$\operatorname{Length}(\partial B_r) = \operatorname{Length}(\partial B) + 2\pi r. \tag{10.2}$$

Theorem 10.2. *Let B be a convex region in ordinary 3-space. Then for all $r \geq 0$*

$$\operatorname{Volume}(B_r) = \operatorname{Volume}(B) + \operatorname{Area}(\partial B)r + \frac{1}{2}k_1(\partial B)r^2 + \frac{4\pi r^3}{3}; \tag{10.3}$$

$$\operatorname{Area}(\partial B_r) = \operatorname{Area}(\partial B) + k_1(\partial B)r + 4\pi r^2. \tag{10.4}$$

[1] Jakob Steiner (1796–1863). Swiss mathematician who was professor at the University of Berlin. Steiner did not learn to read and write until he was 14 and only went to school at the age of 18, against the wishes of his parents. Synthetic geometry was revolutionized by Steiner. He hated analysis as thoroughly as Lagrange hated geometry. He believed that calculation replaces thinking while geometry stimulates thinking.

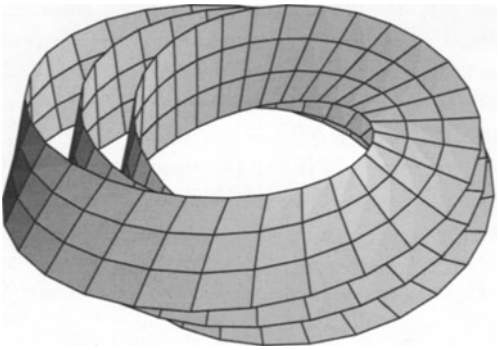
Here, $k_1(\partial B)$ denotes the integral over ∂B of the mean curvature of ∂B; it is defined and generalized below.

Let Vol_k be the function that assigns to each measurable set of a Riemannian manifold its k-dimensional volume. In particular, $\mathrm{Vol}_1 = \text{Length}$, $\mathrm{Vol}_2 = \text{Area}$ and $\mathrm{Vol}_3 = \text{ordinary volume}$. Then just as in Chapter 3 (see Lemma 10.12 below), we have

$$\frac{d}{dr}\mathrm{Vol}_n(B_r) = \mathrm{Vol}_{n-1}(\partial B_r). \tag{10.5}$$

It follows from (10.5) that (10.1) is equivalent to (10.2) and that (10.3) is equivalent to (10.4).

There is also a version of Steiner's Formula that is valid for not necessarily convex regions with smooth boundaries; it is this version that we shall compare with Weyl's Tube Formula. To explain the connection, let M be a complete oriented manifold of dimension n, and let P be a connected hypersurface with compact closure (and of class C^∞). If P is nonorientable, then a tubular hypersurface about P may have only one component, and nothing special can be said about the volumes of tubes about P.



A parallel surface to a Möbius strip is connected.

However, if P is orientable, then each tubular hypersurface P_r has two components, which we label P_r^+ and P_r^-. (See the picture on page 8.) Then P_r^+ and P_r^- are part of the boundary of two tubular regions whose union is the tube of radius r about P. We call these two tubular regions **half-tubes**.

Definition. *For $r \geq 0$ we put*

$$\mathrm{A}_P^{M\pm}(r) = (n-1)\text{-dimensional volume of } P_r^{\pm},$$

$$\mathrm{V}_P^{M\pm}(r) = n\text{-dimensional volume of the portion } T(P,r) \text{ lying between } P_r^{\pm} \text{ and } P.$$

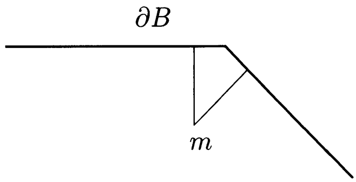
We shall generalize Lemmas 3.13 and 8.3 by showing that

$$\frac{d}{dr}\mathrm{V}_P^{M^\pm}(r) = \mathrm{A}_P^{M^\pm}(r), \tag{10.6}$$

and that (10.6) implies

$$\mathrm{Vol}_n(B_r) = \mathrm{Vol}_n(B) + V_{\partial B}^{M^+}(r). \tag{10.7}$$

In the case that B has a smooth boundary, formulas (10.6) and (10.7) have proofs similar to that of Lemma 3.13. In fact, (10.7) also holds when B is convex and has piecewise smooth boundary (see [Fd1], [Fd2]). But when B is not convex and ∂B is not at least C^1, there are difficulties with defining Fermi coordinates, and hence with (10.7). The problem is that there are certain points m nearby to corners of ∂B such that m has more than one geodesic to ∂B, no matter how close m is to ∂B. Consequently, any tube will overlap itself no matter how small the radius. See [CMS2] and [Fd2] for discussions of this problem and modifications needed in Weyl's Tube Formula to deal with it.



When $M = \mathbb{R}^n$, it is natural to ask if there is an analog of Weyl's Tube Formula for the volume between P and P_r^+ or P_r^-. Such a formula does exist (see Corollary 10.29), and it is equivalent to Steiner's Formula when n is 2 or 3. The main new feature is that only about half the coefficients in the expansion of $V_P^{M\pm}(r)$ are intrinsic; these are multiples of the corresponding coefficients in Weyl's Tube Formula. The remaining coefficients depend on the second fundamental form, but in a mild fashion. Thus the analog of Steiner's Formula can be considered to be a refinement of Weyl's Tube Formula (1.1) that holds for orientable hypersurfaces.

In Section 10.1 we modify the methods of Chapters 2–5, so that they apply specifically to orientable hypersurfaces. The point is that because a tube about an orientable hypersurface has two sides, there are refined versions of many of the comparison theorems of Chapters 2–5. For this we need a refined version ϑ of the infinitesimal volume function that is available for orientable hypersurfaces; it is discussed in Section 10.2. Then in Section 10.3 we study orientable hypersurfaces of complete Riemannian manifolds of nonnegative or nonpositive sectional curvature, and we sharpen the comparison theorems of Sections 8.2–8.5. In Section 10.4 we derive refinements of the comparison theorems of Chapter 8. In particular, we obtain inequalities for the volumes of half-tubes about orientable hypersurfaces in a space with nonnegative or nonpositive sectional curvature. These formulas are specialized in Section 10.5 to prove an n-dimensional generalization of Steiner's Formula.

10.1 Hypersurfaces

Many of the formulas for hypersurfaces that we need are special cases of formulas already derived for general submanifolds. However, the second fundamental form of a hypersurface is so much simpler than that of a general submanifold that it is often easier just to rederive the formulas.

Let M be an orientable Riemannian manifold of dimension n with a fixed orientation. This orientation together with the Riemannian metric of M uniquely specifies a Riemannian volume form ω^M. If P is an orientable hypersurface of M, then each orientation of P together with the induced metric on P gives rise to Riemannian volume forms ω^P and $-\omega^P$. Equivalently, there is a globally defined vector field N normal to P with $\|N\| = 1$. Here, ω^P and N determine each other by the equation

$$\omega^P(X_1, \ldots, X_{n-1}) = \omega^M(X_1, \ldots, X_{n-1}, N) \tag{10.8}$$

for $X_1, \ldots, X_{n-1} \in \mathfrak{X}(P)$. From now on we assume that specific choices of the orientations of M and P have been made, and we say that M and P are **oriented**. Future formulas will depend implicitly on the forms ω^M, ω^P and the vector field N, which are related to one another by (10.8). However, since all the formulas in this section are local, they give information in the nonorientable case as well.

The Riemannian geometry of the embedding of a hypersurface P in M is conveniently described by a variant of the second fundamental form, the **shape operator**. It is defined by

$$SA = -\nabla_A N$$

for $A \in \mathfrak{X}(P)$. It is easy to check that S is a tensor field on P, and that $\langle SA, B\rangle = \langle A, SB\rangle$ for $A, B \in \mathfrak{X}(P)$; thus, S gives rise on each tangent space P_p to a linear transformation that is symmetric with respect to $\langle\ ,\ \rangle$. In the special case of a tubular hypersurface, the shape operator coincides with the operator S that we defined in Chapter 3, and for that reason we use the same notation. In the present chapter we are considering tubular hypersurfaces about a given hypersurface P, so that the shape operator of P is actually $S(0)$ in the notation of Chapter 3; however, we shall continue to write S for $S(0)$.

Let $S^{(1)}, \ldots, S^{(n-1)}$ be the real-valued functions on P defined by the formula

$$\det(I + tS) = \sum_{k=0}^{n-1} S^{(k)} t^k. \tag{10.9}$$

Then $S^{(k)}$ is the k^{th} symmetric polynomial of the eigenvalues of S. In particular, $h = S^{(1)}$ is the ordinary **mean curvature**. For this reason we call $S^{(k)}$ the **mean curvature of order** k.

The Gauss equation (4.28) simplifies considerably when the codimension of P is 1. Explicitly,

$$R^P_{wxyz} - R^M_{wxyz} = \langle Sw, y\rangle\langle Sx, z\rangle - \langle Sw, z\rangle\langle Sx, y\rangle \tag{10.10}$$

for $w, x, y, z \in P_p$ and $p \in P$. The fact that the right-hand side of (10.10) contains only two terms accounts for the fact that computations with the shape operator of a hypersurface are generally much simpler than those for a submanifold of general codimension.

The shape operator S gives rise to a double form $\widetilde{S}$ of type $(1,1)$ via $\widetilde{S}(A)(B) = \langle SA, B \rangle$. To avoid excessive notation, we suppress the tilde. The k^{th} power of S viewed as a double form is given by Lemma 4.1, page 55.

There are simple relations between the S^k's and the $S^{(k)}$'s.

Lemma 10.3. *Let $\{E_1, \ldots, E_{n-1}\}$ be any local orthonormal frame on P. Then for all k we have*

$$S^{(k)} = \frac{1}{(k!)^2} \sum_{a_1 \ldots a_k = 1}^{n-1} S^k(E_{a_1}, \ldots, E_{a_k})(E_{a_1}, \ldots, E_{a_k}). \tag{10.11}$$

Proof. The explicit formula for $S^{(k)}$ (that follows from the definition (10.9) is

$$S^{(k)} = \sum_{1 \le a_1 \le \ldots \le a_k \le n-1} \det(S(E_{a_i})(E_{a_j})). \tag{10.12}$$

Then by problem 4.2 we have

$$S^k(E_{a_1}, \ldots, E_{a_k})(E_{a_1}, \ldots, E_{a_k}) = k! \det(S(E_{a_i})(E_{a_j})). \tag{10.13}$$

From (10.12) and (10.13) follows (10.11). ☐

Corollary 10.4. *The double form S^{2c} is related to the curvature tensors of P and M by the formula*

$$S^{2c} = 2^c (R^P - R^M)^c. \tag{10.14}$$

Proof. The Gauss equation (10.10) can be rewritten in the notation of double forms as

$$S^2 = 2(R^P - R^M). \tag{10.15}$$

Taking the c^{th} powers of both sides of (10.15), we obtain (10.14). ☐

Now we are ready to find the expressions for the $S^{(c)}$'s in terms of curvature. Note that $C^{2c-1}((R^P - R^M)^c)$ can be regarded as a linear transformation, so that $\operatorname{tr}(SC^{2c-1}((R^P - R^M)^c))$ makes sense.

Lemma 10.5. *The higher order mean curvatures are related to the curvature tensors of P and M by the formulas*

$$S^{(2c)} = \frac{2^c}{((2c)!)^2} C^{2c}((R^P - R^M)^c), \tag{10.16}$$

$$S^{(2c+1)} = \frac{2^c}{(2c)!(2c+1)!} \Big(hC^{2c}((R^P - R^M)^c) - 2c \operatorname{tr}\big(SC^{2c-1}((R^P - R^M)^c)\big) \Big). \tag{10.17}$$

Proof. Although (10.16) follows from the proof of Theorems 4.7 and 4.10, we give a direct proof. Let $\{E_1, \dots, E_{n-1}\}$ be a local orthonormal frame on P. It follows from (10.11) and (10.14) that

$$\begin{aligned}
S^{(2c)} &= \frac{1}{((2c)!)^2} \sum_{a_1 \dots a_{2c}=1}^{n-1} S^{2c}(E_{a_1}, \dots, E_{a_{2c}})(E_{a_1}, \dots, E_{a_{2c}}) \\
&= \frac{1}{((2c)!)^2} \sum_{a_1 \dots a_{2c}=1}^{n-1} 2^c (R^P - R^M)^c (E_{a_1}, \dots, E_{a_{2c}})(E_{a_1}, \dots, E_{a_{2c}}) \\
&= \frac{2^c}{((2c)!)^2} C^{2c}((R^P - R^M)^c),
\end{aligned}$$

proving (10.16).

To prove (10.17), we first note that

$$\begin{aligned}
S^{2c+1}(E_{a_1}, \dots, E_{a_{2c+1}})(E_{a_1}, \dots, E_{a_{2c+1}}) &= \sum_{ij=1}^{2c+1} (-1)^{i+j} S(E_{a_i})(E_{a_j}) \\
&\quad \cdot S^{2c}(E_{a_1}, \dots, \hat{E}_{a_i}, \dots, E_{a_{2c+1}})(E_{a_1}, \dots, \hat{E}_{a_j}, \dots, E_{a_{2c+1}}) \\
&= \textstyle\sum_I + \sum_{II},
\end{aligned}$$

where

$$\textstyle\sum_I = \displaystyle\sum_{i=1}^{2c+1} S(E_{a_i})(E_{a_i}) S^{2c}(E_{a_1}, \dots, \hat{E}_{a_i}, \dots, E_{a_{2c+1}}) \cdot (E_{a_1}, \dots, \hat{E}_{a_i}, \dots, E_{a_{2c+1}}) \tag{10.18}$$

and

$$\begin{aligned}
\textstyle\sum_{II} &= 2 \sum_{1 \le i < j \le 2c+1} (-1)^{i+j} S(E_{a_i})(E_{a_j}) S^{2c}(E_{a_1}, \dots, \hat{E}_{a_i}, \dots, E_{a_{2c+1}}) \\
&\quad \cdot (E_{a_1}, \dots, \hat{E}_{a_j}, \dots, E_{a_{2c+1}}).
\end{aligned}$$

It follows from (10.14) that

$$\sum_{a_1 \dots a_{2c+1}=1}^{n-1} \textstyle\sum_I = 2^c (2c+1) h \, C^{2c}((R^P - R^M)^c). \tag{10.19}$$

The summation of $\sum_{II}$ is more complicated. Using (10.14) we calculate

$$\begin{aligned}
\sum_{a_1 \dots a_{2c+1}=1}^{n-1} \textstyle\sum_{II} &= \displaystyle\sum_{a_1 \dots a_{2c+1}=1}^{n-1} \Bigg(\sum_{1 \le i < j \le 2c+1} (-1)^{i+j} S(E_{a_i})(E_{a_j}) \\
&\quad \cdot 2^{c+1} (R^P - R^M)^c (E_{a_1}, \dots, \hat{E}_{a_i}, \dots, E_{a_{2c+1}}) \cdot (E_{a_1}, \dots, \hat{E}_{a_j}, \dots, E_{a_{2c+1}}) \Bigg)
\end{aligned} \tag{10.20}$$

$$= -2^{c+1} \sum_{a_1\ldots a_{2c+1}=1}^{n-1} \Bigg(\sum_{1\le i<j\le 2c+1} S(E_{a_i})(E_{a_j}) \\ \cdot(R^P - R^M)^c(E_{a_j}, E_{a_1}, \ldots, \hat{E}_{a_i}, \ldots, \hat{E}_{a_j}, \ldots, E_{a_{2c+1}}) \\ \cdot(E_{a_i}, E_{a_1}, \ldots, \hat{E}_{a_i}, \ldots, \hat{E}_{a_j}, \ldots, E_{a_{2c+1}}) \Bigg).$$

Here,

$$\begin{aligned} \sum_{a_1\ldots\hat{a}_i\ldots\hat{a}_j\ldots a_{2c+1}=1}^{n-1} & (R^P - R^M)^c(E_{a_j}, E_{a_1}, \ldots, \hat{E}_{a_i}, \ldots, \hat{E}_{a_j}, \ldots, E_{a_{2c+1}}) \\ & \cdot (E_{a_i}, E_{a_1}, \ldots, \hat{E}_{a_i}, \ldots, \hat{E}_{a_j}, \ldots, E_{a_{2c+1}}) \\ = \ & C^{2c-1}((R^P - R^M)^c)(E_{a_i})(E_{a_j}), \end{aligned}$$

and so (10.20) reduces to

$$\begin{aligned} \sum_{a_1\ldots a_{2c+1}=1}^{n-1} \Sigma_{II} &= -2^{2c+1} \sum_{1\le i<j\le 2c+1} \sum_{a_i a_j=1}^{n-1} S(E_{a_i})(E_{a_j}) \\ & \qquad \cdot C^{2c-1}((R^P - R^M)^c)(E_{a_j})(E_{a_i}) \\ &= -2^c(2c)(2c+1)\operatorname{tr}\big(SC^{2c-1}((R^P - R^M)^c)\big). \end{aligned} \tag{10.21}$$

From (10.19) and (10.21) we get (10.17). □

Thus we see that each $S^{(2c)}$ can be expressed entirely in terms of the curvature tensor of P. This is not true for the $S^{(2c+1)}$'s; however, each of them depends only linearly on S.

Definition. *Suppose that P is an oriented hypersurface with compact closure in an orientable Riemannian manifold M. Let R be any tensor field on P having the same type and symmetries as the curvature tensor R^P, and let L be any $(1,1)$ tensor field on P. We put*

$$k_{2c}(R) = \frac{1}{c!(2c)!} \int_P C^{2c}(R^c)\omega^P \tag{10.22}$$

$$\begin{aligned} k_{2c+1}(R, L) = \frac{1}{c!(2c)!} \int_P & \{\operatorname{tr}(L)C^{2c}(R^c) \\ & - 2c\operatorname{tr}(LC^{2c-1}(R^c))\}\omega^P. \end{aligned} \tag{10.23}$$

We call $k_{2c}(R^P - R^M)$ and $k_{2c+1}(R^P - R^M, S)$ the $(2c)^{\text{th}}$ and $(2c+1)^{\text{th}}$ **integrated mean curvatures** *of P in M.*

(Equation (10.22) coincides with (4.4) of Chapter 4, but (10.23) is new. Notice that (10.23) makes sense for any codimension, although we need it only in the codimension 1 case.)

Lemma 10.6. *Let P be a topologically embedded oriented hypersurface with compact closure in an oriented Riemannian manifold M. Then the integrals of the $S^{(c)}$'s are related to the integrated mean curvatures by the formulas*

$$\int_P S^{(2c)}\omega^P = \frac{k_{2c}(R^P - R^M)}{1\cdot 3\cdots(2c-1)}, \tag{10.24}$$

$$\int_P S^{(2c+1)}\omega^P = \frac{k_{2c+1}(R^P - R^M, S)}{1\cdot 3\cdots(2c+1)}. \tag{10.25}$$

Proof. We do (10.25); the proof of (10.24) is simpler. From (10.17) and (10.23) we have that

$$\begin{aligned}\int_P S^{(2c+1)}\omega^P &= \frac{2^c}{(2c)!(2c+1)!}\int_P \Big(hC^{2c}((R^P - R^M)^c) \\ &\qquad - 2c\,\mathrm{tr}\big(SC^{2c-1}((R^P - R^M)^c)\big)\Big)\omega^P \\ &= \frac{2^c c!}{(2c+1)!}k_{2c+1}(R^P - R^M, S) \\ &= \frac{k_{2c+1}(R^P - R^M, S)}{1\cdot 3\cdots(2c+1)}.\end{aligned}$$

□

Thus we get a refinement for orientable hypersurfaces of Theorem 4.10:

Corollary 10.7. *We have*

$$\int_P \det(I \pm tS)\omega^P = \sum_{c=0}^{\left[\frac{n-1}{2}\right]} \frac{k_{2c}(R^P - R^M)t^{2c}}{1\cdot 3\cdots(2c-1)} \pm \sum_{c=0}^{\left[\frac{n}{2}-1\right]} \frac{k_{2c+1}(R^P - R^M, S)t^{2c+1}}{1\cdot 3\cdots(2c+1)}.$$

Proof. This follows from (10.9), (10.24) and (10.25). □

10.2 The Infinitesimal Change of Volume Function of a Hypersurface

In this section P will be an oriented hypersurface with compact closure in a complete oriented Riemannian manifold M. Let $(x_1, \ldots, x_n)$ be a system of Fermi coordinates for P at $p \in P$. Then the restrictions of $x_1, \ldots, x_{n-1}$ to P form a coordinate system on P, while x_n measures the distance from P in M. It will be

assumed that $(x_1, \ldots, x_n)$ is oriented coherently with P and M. Then the definitions (2.20) of the function σ and the unit normal vector field N in terms of Fermi coordinates reduce to

$$\sigma = x_n \qquad \text{and} \qquad N = \frac{\partial}{\partial x_n}.$$

Previously, σ was a distance function, but now we allow it to be negative when x_n is.

Suppose now that the oriented system of Fermi coordinates is centered at $p \in P$. Let $t \longmapsto \xi(t)$ be the geodesic in M with $\xi(0) = p$ and $\xi'(0) = N_p$. Put

$$\vartheta(t) = \omega^M\left(\frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_n}\right)(\xi(t)). \tag{10.26}$$

It is easy to check that ϑ does not depend on the choice of oriented Fermi coordinates at p. The function ϑ is a variant of the infinitesimal volume function ϑ_u first defined in Chapter 3; in fact,

$$\vartheta(t) = (\text{chvol} \circ \exp_\nu)(p, tN_p) = \vartheta_{N_p}(t). \tag{10.27}$$

(For example, when $M = \mathbb{R}^n$ it turns out that $\vartheta(t) = \det(I - tS)$.) In the case of an oriented hypersurface the function $\vartheta(t)$ is defined for negative as well as positive t, whereas for a general submanifold the infinitesimal change of volume function $\vartheta_u(t)$ is only defined for nonnegative t. Explicit formulas are given in the next lemma.

Lemma 10.8. *For a unit normal N to the hypersurface P, let S_N denote the shape operator defined using N (so $S = S_N$). Then for $p \in P$ we have*

$$\vartheta(-t) = \vartheta_{-N_p}(t), \tag{10.28}$$

and

$$S_{N_p}(-t) = -S_{-N_p}(t). \tag{10.29}$$

Proof. Equation (10.28) is obvious from (10.27), and (10.29) follows from the definition of S. □

This explains why it is unnecessary to make explicit the normal direction in the definition of ϑ.

Lemma 10.9. *For $-e_c(p, -N_p) \le t \le e_c(p, N_p)$, we have*

$$\frac{\vartheta'(t)}{\vartheta(t)} = -\operatorname{tr} S(t). \tag{10.30}$$

Proof. For $t \geq 0$ equation (10.30) is a special case of Theorem 3.11. (In fact, the proof of (10.30) is considerably simpler than that of Theorem 3.11.) Furthermore, let $t > 0$. By the chain rule and (10.28) we have

$$\vartheta'(-t) = -\vartheta'_{-N_p}(t).$$

From this observation and (10.29) we get (10.30) for negative as well as positive t. □

We define Θ in the same way that we defined Θ_u in Chapter 8:

$$\Theta(t) = \begin{cases} \vartheta(t) & \text{for } -e_c(p, -N_p) < t < e_c(p, N_p), \\ 0 & \text{otherwise.} \end{cases}$$

Next we obtain the improved version of Lemma 8.3 that is available for smooth hypersurfaces.

Lemma 10.10. *Let P be a topologically embedded oriented hypersurface with compact closure in an oriented Riemannian manifold M. Then for all $r \geq 0$ we have*

$$\frac{d}{dr}\mathrm{V}_P^{M^\pm}(r) = \mathrm{A}_P^{M^\pm}(r) = \int_P \Theta(\pm r)\omega^P.$$

The proof of Lemma 10.10 is very similar to that of Lemmas 3.12, 3.13, 8.2 and 8.3. Furthermore, the proof of the following lemma is an easy consequence of the definitions.

Lemma 10.11. *For r not greater than the distance from P to its nearest cut-focal point we have*

$$\begin{aligned} \mathrm{A}_P^M(r) &= \mathrm{A}_P^{M^+}(r) + \mathrm{A}_P^{M^-}(r), \\ \mathrm{V}_P^M(r) &= \mathrm{V}_P^{M^+}(r) + \mathrm{V}_P^{M^-}(r), \\ \mathrm{A}_P^{M^-}(r) &= \mathrm{A}_P^{M^+}(-r). \end{aligned}$$

Next we relate the functions Vol_n with the tubular volumes.

Lemma 10.12. *Let B be an open set with smooth boundary and compact closure in an oriented Riemannian manifold M. Then for all $r \geq 0$ we have*

$$\mathrm{Vol}_n(B_r) = \mathrm{Vol}_n(B) + V_{\partial B}^{M^+}(r); \tag{10.31}$$

$$\frac{d}{dr}\mathrm{Vol}_n(B_r) = \mathrm{Vol}_{n-1}(\partial B_r), \tag{10.32}$$

Proof. Equation (10.31) is a consequence of the additivity of the volume function of M. Furthermore, it follows from (10.31) and Lemma 10.10 that

$$\frac{d}{dr}\mathrm{Vol}_n(B_r) = \frac{d}{dr}\left(V_{\partial B}^{M^+}(r)\right) = A_{\partial B}^{M^+}(r) = \mathrm{Vol}_{n-1}(\partial B_r),$$

so we get (10.32). □

10.3 Hypersurfaces in Manifolds of Nonnegative or Nonpositive Curvature

There is a refinement for hypersurfaces of the generalization of Weyl's Tube Formula that we gave in Section 8.2. We shall assume throughout this section that P is an oriented hypersurface with compact closure in a complete oriented n-dimensional Riemannian manifold M.

For $p \in P$ we define

$$\kappa = \min\{\, \langle Sx, x\rangle \;\mid\; x \in P_p,\ \|x\| = 1\,\}.$$

The following lemma is almost a special case of Lemma 8.10.

Lemma 10.13. *Assume that $K^M \geq 0$ and that $S(t)$ is defined for $-\epsilon_1 \leq t \leq \epsilon_2$. Then*

$$\max\{\,\epsilon_1, \epsilon_2\,\}\kappa \leq 1, \tag{10.33}$$

and on $(-\epsilon_1, \epsilon_2)$ we have

$$\operatorname{tr} S(t) \;\geq\; \operatorname{tr}\left(\frac{S}{I - t\,S}\right) \quad \textit{for } 0 \leq t \leq \epsilon_2, \tag{10.34}$$

$$\operatorname{tr} S(t) \;\leq\; \operatorname{tr}\left(\frac{S}{I - t\,S}\right) \quad \textit{for } -\epsilon_1 \leq t \leq 0. \tag{10.35}$$

Proof. Equation (10.34)) is a special case of (8.13). Then (10.35)) follows from (10.34) and (10.29). Explicitly, for $t \leq 0$ we have

$$\operatorname{tr} S(t) = -\operatorname{tr} S_{-N_p}(-t) \leq -\operatorname{tr}\left(\frac{S_{-N_p}}{I + tS_{-N_p}}\right) = \operatorname{tr}\left(\frac{S}{I - tS}\right). \qquad \square$$

Next we obtain a sharpening of Lemmas 8.12 and 3.14.

Lemma 10.14. *Assume that $K^M \geq 0$.*

(i) *For $t \geq 0$*

$$t \;\longmapsto\; \frac{\Theta(t)}{\max(\det(I - t\,S), 0)}$$

is a nonincreasing function, and for t ≤ 0 it is a nondecreasing function.

(ii) *For all* $t \geq 0$

$$\Theta(t) \leq \max(\det(I - t\,S), 0) \leq \max\left(\left(1 - \frac{t\,h}{n-1}\right)^q, 0\right). \tag{10.36}$$

(iii) *In the range* $-e_c(p, -N_p) \leq t \leq e_c(p, N_p)$ *we have*

$$\Theta(t) = \vartheta(t) \leq \det(I - t\,S) \leq \left(1 - \frac{t\,h}{n-1}\right)^{n-1}, \tag{10.37}$$

so that the first zero of $\det(I - t\,S)$ *in either direction does not occur before the first zero of* $\Theta(t)$.

Proof. We prove (i) for the case when $t \leq 0$. From Lemma 10.9 and (10.35) we have

$$\frac{d}{dt}\log\Theta(t) = \frac{\vartheta'(t)}{\vartheta(t)} = -\operatorname{tr} S(t) \geq -\operatorname{tr}\left(\frac{S}{I - tS}\right) = \frac{d}{dt}\log(\det(I - tS)).$$

Just as in the proof of Lemma 8.12, we get that

$$t \longmapsto \frac{\Theta(t)}{\max(\det(I - t\,S), 0)}$$

is nondecreasing for $t \leq 0$.

The proofs of (ii) and (iii) are similar to the proof of Lemma 8.12. $\square$

There are analogous results in the nonpositive curvature case.

Lemma 10.15. *Assume that* $K^M \leq 0$ *and that* $S(t)$ *and its eigenvectors are defined and differentiable for* $-\epsilon_1 \leq t \leq \epsilon_2$, *and that* (10.33) *holds. Then on* $(-\epsilon_1, \epsilon_2)$ *we have*

$$\begin{cases} \operatorname{tr} S(t) \;\leq\; \operatorname{tr}\left(\dfrac{S}{I - t\,S}\right) & \text{for } 0 \leq t \leq \epsilon_2, \\[2ex] \operatorname{tr} S(t) \;\geq\; \operatorname{tr}\left(\dfrac{S}{I - t\,S}\right) & \text{for } -\epsilon_1 \leq t \leq 0. \end{cases} \tag{10.38}$$

Lemma 10.16. *Suppose* $K^M \leq 0$. *Then for* $-e_c(p, -N_p) < t < e_c(p, N_p)$

$$t \longmapsto \frac{\Theta(t)}{\det(I - t\,S)}$$

is a nondecreasing function for $t \geq 0$ *and a nonincreasing function for* $t \leq 0$. *Furthermore,*

$$\Theta(t) = \vartheta(t) \geq \det(I - t\,S).$$

Combining Lemmas 10.13–10.16, we obtain

Corollary 10.17. *For Euclidean space* $\mathbb{R}^n$ *we have*

$$\operatorname{tr} S(t) = \operatorname{tr}\left(\frac{S}{I - t\,S}\right)$$

and

$$\Theta(t) = \vartheta(t) = \det(I - tS)$$

for $-e_c(p, -N_p) < t < e_c(p, N_p)$.

If instead of assuming a lower bound on the sectional curvature of M we assume a lower bound on the Ricci curvature ρ^M, it is still possible to estimate $S(t)$ and the tube volumes, provided that P is a hypersurface. The resulting inequalities are weaker than those of Lemma 10.14 and involve the mean curvature of the submanifold P.

Lemma 10.18. *Suppose the Ricci curvature of* M *is nonnegative. Assume that* $S(t)$ *is defined for* $-\epsilon_1 \le t \le \epsilon_2$. *Then*

$$\max\{\,\epsilon_1, \epsilon_2\,\}h \le n - 1, \tag{10.39}$$

and on $(0, t_1(u))$ *we have*

$$\operatorname{tr} S(t) \;\ge\; h\left(1 - \frac{t\,h}{n-1}\right)^{-1} \quad \textit{for } 0 \le t \le \epsilon_2, \tag{10.40}$$

$$\operatorname{tr} S(t) \;\le\; h\left(1 - \frac{t\,h}{n-1}\right)^{-1} \quad \textit{for } -\epsilon_1 \le t \le 0. \tag{10.41}$$

Proof. P is an orientable hypersurface. Let

$$f(t) = \frac{\operatorname{tr} S(t)}{n-1};$$

Theorem 3.19 (using the Cauchy-Schwarz Inequality) yields

$$f'(t) \ge f(t)^2.$$

Since $\dim P = n - 1$, we have

$$h = \operatorname{tr} S(0) = (n-1)f(0).$$

From Lemma 8.7 it follows that (10.39) holds, and that

$$\frac{\operatorname{tr} S(t)}{n-1} = f(0) \ge \frac{f(0)}{1 - tf(0)} = \frac{\dfrac{h}{n-1}}{1 - \dfrac{t\,h}{n-1}}. \tag{10.42}$$

It also follows from Lemma 8.7 that the denominator on the right-hand side of (10.42) does not vanish. Thus we get (10.40). Then (10.41) follows from (10.40) and (10.29). ∎

Lemma 10.19. *Assume that M has nonnegative Ricci curvature.*

(i) *The map*

$$t \longmapsto \frac{\Theta(t)}{\max\left(\left(1-\dfrac{t\,h}{n-1}\right)^{n-1},0\right)} \tag{10.43}$$

is nonincreasing for $t \geq 0$, and nondecreasing for $t \leq 0$.

(ii) *For all $t \geq 0$*

$$\Theta(t) \leq \max\left(\left(1-\frac{t\,h}{n-1}\right)^{n-1},0\right). \tag{10.44}$$

(iii) *In the range $-e_c(p,-N_p) \leq t \leq e_c(p,N_p)$ we have*

$$\Theta(t) = \vartheta(t) \leq \det(I - t\,S) \leq \left(1-\frac{t\,h}{n-1}\right)^{n-1}, \tag{10.45}$$

so that the first zero of $\det(I - t\,S)$ *in either direction does not occur before the first zero of* $\Theta(t)$.

Proof. First, suppose $t \geq 0$ and let $t_1 = \inf\{t \mid 0 \leq t\,h < n-1\}$. We use Theorem 3.11 in conjunction with (10.40) to obtain

$$\frac{d}{dt}\log\vartheta(t) = \frac{\vartheta'(t)}{\vartheta(t)} = -\operatorname{tr} S(t) \leq \frac{-h}{1-\dfrac{t\,h}{n-1}} = \frac{d}{dt}\log\left(1-\frac{t\,h}{n-1}\right)^{n-1},$$

and so

$$t \longmapsto \frac{\vartheta(t)}{\left(1-\dfrac{t\,h}{n-1}\right)^{n-1}}$$

is nonincreasing. Since $\vartheta(0) = 1$, we get (10.43) and (10.44) under the assumption that $0 \leq t_1$. From the definitions we have

$$t_1 \geq e_f(p,N_p) \geq e_c(p,N_p).$$

Since $\Theta(t) = 0$ for $t \geq e_c(N_p,p)$, it follows that (10.43) and (10.44) hold for all $t \geq 0$. The proofs of (10.43) and (10.44) for $t \leq 0$ are similar. $\square$

10.4 Steiner's Formula for a Space of Nonnegative or Nonpositive Curvature

Our main task in this section is to establish an n-dimensional version of Steiner's Formula for a region B in $\mathbb{R}^n$ with smooth boundary ∂B. As we have observed in Lemma 10.12, this is equivalent to computing $V_{\partial B}^{\mathbb{R}^n \pm}(r)$. However, because we have the techniques of Chapter 8 at our disposal, it is not much more difficult to prove theorems about a complete manifold M of nonnegative or nonpositive sectional curvature instead of $\mathbb{R}^n$. Therefore, in this section we assume that M is a complete n-dimensional Riemannian manifold and that P is a topologically embedded hypersurface with compact closure.

Theorem 10.20. *Suppose* $K^M \geq 0$.

(i) *For* $0 \leq r \leq \operatorname{minfoc}(P)$ *we have*

$$V_P^{M\pm}(r) \leq \sum_{c=0}^{[\frac{n-1}{2}]} \frac{k_{2c}(R^P - R^M) r^{2c+1}}{1 \cdot 3 \cdots (2c+1)} \mp \sum_{c=0}^{[\frac{n}{2}-1]} \frac{k_{2c+1}(R^P - R^M) r^{2c+2}}{1 \cdot 3 \cdots (2c+1)(2c+2)}, \tag{10.46}$$

(ii) *For all* $r \geq 0$

$$\begin{aligned} V_P^{M\pm}(r) &\leq \int_0^r \int_P \max(\det(I \mp tS), 0)\, dP\, dt \\ &\leq \int_0^r \int_P \max\left(\left(1 \mp \frac{t\,h}{n-1}\right)^{n-1}, 0\right) dP\, dt. \end{aligned} \tag{10.47}$$

Proof. Equation (10.47) follows from Lemma 10.10 and (10.36). Then (10.46) follows from (10.47) and Corollary 10.7. □

The corresponding result in the nonpositive curvature case is:

Theorem 10.21. *If* $K^M \leq 0$, *then for* $0 \leq r \leq \operatorname{minfoc}(P)$

$$V_P^{M\pm}(r) \geq \int_0^r \int_P \det(I \mp tS)\, dP\, dt \tag{10.48}$$

$$= \sum_{c=0}^{[\frac{n-1}{2}]} \frac{k_{2c}(R^P - R^M) r^{2c+1}}{1 \cdot 3 \cdots (2c+1)} \mp \sum_{c=0}^{[\frac{n}{2}-1]} \frac{k_{2c+1}(R^P - R^M) r^{2c+2}}{1 \cdot 3 \cdots (2c+1)(2c+2)}.$$

Combining Theorems 10.20 and 10.21, we get

Corollary 10.22. *For Euclidean space $\mathbb{R}^n$ we have*

$$V_P^{\mathbb{R}^n\pm}(r) = \int_0^r \int_{P} \det(I \mp tS)\, dP\, dt$$

$$= \sum_{c=0}^{[\frac{n-1}{2}]} \frac{k_{2c}(R^P) r^{2c+1}}{1\cdot 3\cdots(2c+1)} \mp \sum_{c=0}^{[\frac{n}{2}-1]} \frac{k_{2c+1}(R^P) r^{2c+2}}{1\cdot 3\cdots(2c+1)(2c+2)}$$

for $0 \le r \le \operatorname{minfoc}(P)$. □

There are tube volume estimates involving the Ricci curvature that are weaker than those of Theorem 10.20:

Theorem 10.23. *Assume that the Ricci curvature of M is nonnegative.*

(i) *For* $0 \le r \le \operatorname{minfoc}(P)$ *we have*

$$V_P^{M\pm}(r) \le \frac{r}{n} \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n}{2k+1} \left(\frac{r}{n-1}\right)^{2k} \int_P h^{2k}\, dP$$

$$\mp \frac{r}{n} \sum_{k=0}^{[\frac{n}{2}]-1} \binom{n}{2k+2} \left(\frac{r}{n-1}\right)^{2k+1} \int_P h^{2k}\, dP.$$

(ii) *For all* $r \ge 0$ *we have*

$$V_P^{M\pm}(r) \le \int_0^r \int_P \max\left(\left(1 \mp \frac{t\,h}{n-1}\right)^{n-1}, 0\right) dP\, dt.$$

Instead of giving the proof of this theorem, we prove its corollary:

Corollary 10.24. *Assume that the Ricci curvature of M is nonnegative. Then for* $0 \le r \le \operatorname{minfoc}(P)$

$$V_P^M(r) \le \frac{2r}{n} \sum_{k=0}^{[\frac{n-1}{2}]} \binom{n}{2k+1} \left(\frac{r}{n-1}\right)^{2k} \int_P h^{2k}\, dP. \tag{10.49}$$

Proof. We have

$$\int_{S^0(1)} \vartheta_u(r)\, du = \vartheta_u(r) + \vartheta_{-u}(r) = \vartheta_u(r) + \vartheta_u(-r),$$

and so (10.44) implies that

$$\int_{S^0(1)} \vartheta_u(r)\, du \le \left(1 - \frac{r\,h}{n-1}\right)^{n-1} + \left(1 + \frac{r\,h}{n-1}\right)^{n-1}.$$

Therefore, in the case that P is orientable we have

$$\begin{aligned} \mathrm{A}_P^M(r) &= \int_P \int_{S^0(1)} \vartheta_u(r)\, du\, dP \qquad (10.50) \\ &\le \int_P \left\{ \left(1 - \frac{r\,h}{n-1}\right)^{n-1} + \left(1 + \frac{r\,h}{n-1}\right)^{n-1} \right\} dP. \end{aligned}$$

When (10.50) is integrated from 0 to r and the right-hand side of (10.50) is expanded, we get (10.49) for an orientable hypersurface P.

We can divide up a nonorientable hypersurface P into orientable pieces. Then (10.49) holds for each piece. Since (10.49) does not depend on the choice of orientation and integration over P is additive, we see that (10.49) holds for all of P. □

Corollary 10.25. *Assume in Theorem 10.24 that P is a minimal hypersurface. Then for $0 \le r \le \operatorname{minfoc}(P)$ we have*

$$\mathrm{V}_P^{M^\pm}(r) \le r \text{ volume}(P)$$

and

$$\mathrm{V}_P^{M}(r) \le 2r \text{ volume}(P).$$

Proof. When $h = 0$, the right-hand side of (10.49) reduces to one term. □

10.5 Inequalities that Generalize Steiner's Formula

To obtain generalizations of Steiner's Formula, it is not necessary to suppose that the ambient space M be flat. Instead we assume that the sectional curvature of M satisfies $K^M \ge 0$ or $K^M \le 0$. Furthermore, it will not be necessary to assume that the region B is convex, but we do suppose that the boundary of B is smooth. The case that $\dim M \le 3$ is especially interesting because the estimates can be written in terms of the Euler characteristic of B or ∂B.

Theorem 10.26. *Let B be a region in a complete surface M. Suppose that every point p with* $\operatorname{distance}(p, B) \le r$ *has a unique shortest geodesic from it to B.*

(i) *If $K^M \ge 0$, then*

$$\begin{aligned} \operatorname{Area}(B_r) &\le \operatorname{Area}(B) + \operatorname{Length}(\partial B) r \qquad (10.51) \\ &\quad + \pi \left\{ \chi(B) - \frac{1}{2} \int_B K^M dB \right\} r^2; \\ \operatorname{Length}(\partial B_r) &\le \operatorname{Length}(\partial B) + 2\pi \left\{ \chi(B) - \frac{1}{2} \int_B K^M dB \right\} r. \qquad (10.52) \end{aligned}$$

(ii) *If $K^M \le 0$, then the inequalities in* (10.51) *and* (10.52) *are reversed.*

Proof. It suffices to prove (10.51). In dimension 2 equation (10.31) becomes

$$\mathrm{Area}(B_r) = \mathrm{Area}(B) + V_{\partial B}^{M^+}(r). \tag{10.53}$$

Furthermore, (10.46) specializes to

$$V_{\partial B}^{M^+}(r) \leq \mathrm{Length}(\partial B)r - \frac{1}{2}k_1(R^{\partial B} - R^M, S)r^2. \tag{10.54}$$

Now the Gauss-Bonnet Theorem for a surface with boundary (for example see [ON1, page 375]) implies that

$$k_1(R^{\partial B} - R^M, S)r^2 = \int_{\partial B} \mathrm{tr}(S)\omega^{\partial B} = -2\pi\chi(B) + \int_B K^M dB. \tag{10.55}$$

Then from (10.53)–(10.55) we get (10.51). □

Theorem 10.27. *Let B be a region in a complete 3-dimensional Riemannian manifold M. Denote by $K^M|\partial B$ the restriction of K^M to the plane sections tangent to ∂B.*

(i) *If $K^M \geq 0$, then*

$$\begin{aligned} \mathrm{Vol}(B_r) \;&\leq\; \mathrm{Vol}(B) + \mathrm{Area}(\partial B)r + \frac{1}{2}\left(\int_{\partial B} h\, d(\partial B)\right) r^2 \\ &\quad + \frac{1}{3}\left\{2\pi\chi(B) - \int_{\partial B} (K^M|_{\partial B}) d(\partial B)\right\} r^3; \end{aligned} \tag{10.56}$$

$$\begin{aligned} \mathrm{Area}(\partial B_r) \;&\leq\; \mathrm{Area}\left(\int_{\partial B} h\, dB\right) r \\ &\quad + \left\{2\pi\chi(\partial B) - \int_{\partial B} (K^M|_{\partial B}) d(\partial B)\right\} r^2. \end{aligned} \tag{10.57}$$

(ii) *If $K^M \leq 0$, then the inequalities in* (10.56) *and* (10.57) *are reversed.*

Proof. The calculations are similar to those of Theorem 10.26. From (10.46) we have

$$\begin{aligned} V_{\partial B}^{M^+}(r) \;&\leq\; \mathrm{Area}(\partial B)r - \frac{1}{2}k_1(R^{\partial B} - R^M, S)r^2 \\ &\quad + \frac{1}{3}k_2(R^{\partial B} - R^M)r^3. \end{aligned} \tag{10.58}$$

But

$$k_1(R^{\partial B} - R^M, S) = \int_{\partial B} h\, d(\partial B) \tag{10.59}$$

and

$$k_2(R^{\partial B} - R^M) = 2\pi\chi(B) - \int_{\partial B} (K^M|_{\partial B}) d(\partial B). \tag{10.60}$$

From (10.31) and (10.58)–(10.60) we get (10.56), and hence also (10.57). □

The n-dimensional generalization of Theorems 10.26 and 10.27 is as follows.

Theorem 10.28. *Suppose that M is a complete n-dimensional Riemannian manifold and that B is a region in M.*

(i) *If $K^M \geq 0$, then*

$$\mathrm{Vol}_n(B_r) \leq \mathrm{Vol}_n(B) + \sum_{c=0}^{[(n-1)/2]} \frac{k_{2c}(R^P - R^M) r^{2c+1}}{1 \cdot 3 \cdots (2c+1)} - \sum_{c=0}^{[n/2]-1} \frac{k_{2c+1}(R^P - R^M, S) r^{2c+2}}{1 \cdot 3 \cdots (2c+1)(2c+2)}. \tag{10.61}$$

(ii) *If $K^M \leq 0$, then the inequality is reversed in* (10.61).

Proof. Part (i) is an immediate consequence of (10.31) and (10.46), and part (ii) follows from (10.31) and (10.47). □

Corollary 10.29. *Suppose B is a region in $\mathbb{R}^n$. Then*

$$\mathrm{Vol}_n(B_r) = \mathrm{Vol}_n(B) + \sum_{c=0}^{[(n-1)/2]} \frac{k_{2c}(R^P) r^{2c+1}}{1 \cdot 3 \cdots (2c+1)} - \sum_{c=0}^{[n/2]-1} \frac{k_{2c+1}(R^P, S) r^{2c+2}}{1 \cdot 3 \cdots (2c+1)(2c+2)}.$$

10.6 Problems

10.1 Show that the volume of a half-tube about an orientable hypersurface P in a space $\mathbb{K}^n(\lambda)$ of constant sectional curvature λ is given by

$$\begin{aligned}
\frac{d}{dr} V_P^{\mathbb{K}^n(\lambda)^{\pm}}(r) &= \int_P \det\left(\cos(r\sqrt{\lambda}) I \mp \frac{\sin(r\sqrt{\lambda})}{\sqrt{\lambda}} S\right) dP \\
&= (\cos(r\sqrt{\lambda}))^{n-1} \left\{ \sum_{c=0}^{[\frac{1}{2}(n-1)]} \frac{k_{2c}(R^P - R^{\mathbb{K}^n(\lambda)})}{1 \cdot 3 \cdots (2c-1)} \left(\frac{\tan(r\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{2c} \right. \\
&\quad \left. \mp \sum_{c=0}^{[\frac{1}{2}n-1]} \frac{k_{2c+1}(R^P - R^{\mathbb{K}^n(\lambda)}, S)}{1 \cdot 3 \cdots (2c+1)} \left(\frac{\tan(r\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{2c+1} \right\}.
\end{aligned}$$

10.2 Prove that the infinitesimal change of volume function of an orientable hypersurface P of an orientable Riemannian manifold M has the power series expansion in Fermi coordinates that starts off as

$$\vartheta(r) = \Big\{1 - \langle H, N\rangle r + \frac{1}{2}\left(\rho^M_{NN} + \tau^P - \tau^M\right) r^2$$

$$+\frac{1}{6}\left(-\nabla_N(\rho^M)_{NN} + \langle H, N\rangle\left(\rho^M_{NN} - \tau^P + \tau^M\right) + 2\langle S, \rho^P - \rho^M\rangle\right) r^3$$

$$+\text{higher order terms}\Big\}_m .$$

10.3 Let P be an orientable hypersurface in a complete orientable Riemannian manifold M. Define

$$\mathrm{B}_P^{M^\pm}(r,\lambda) = \sum_{c=0}^{[(n-1)/2]} \frac{k_{2c}(R^P - R^M)}{1\cdot 3\cdots(2c-1)}(\cos(r\sqrt{\lambda}))^{n-1-2c}\left(\frac{\sin(r\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{2c}$$
$$\mp \sum_{c=0}^{[n/2]-1} \frac{k_{2c+1}(R^P - R^M, S)}{1\cdot 3\cdots(2c+1)}(\cos(r\sqrt{\lambda}))^{n-2c}\left(\frac{\sin(r\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{2c+1}$$

and

$$\mathrm{C}_P^{M^\pm}(r,\lambda) = \sum_{c=0}^{[(n-1)/2]} \binom{n-1}{2c}(\cos(r\sqrt{\lambda}))^{n-1-2c} \cdot \left(\frac{\sin(r\sqrt{\lambda})}{(n-1)\sqrt{\lambda}}\right)^{2c}\int_P h^{2c}dP$$
$$\mp \sum_{c=0}^{[n/2]-1} \binom{n-1}{2c+1}(\cos(r\sqrt{\lambda}))^{n-2c} \cdot \left(\frac{\sin(r\sqrt{\lambda})}{(n-1)\sqrt{\lambda}}\right)^{2c+1}\int_P h^{2c+1}dP.$$

Suppose that $0 \le r \le \operatorname{minfoc}(P)$. Show that

(i) $K^M \ge \lambda$ implies

$$\mathrm{A}_P^{M^\pm}(r) \le \mathrm{B}_P^{M^\pm}(r,\lambda) \le \mathrm{C}_P^{M^\pm}(r,\lambda);$$

(ii) $\rho^M(x,x) \ge \lambda\|x\|^2$ for all tangent vectors x to M implies

$$\mathrm{A}_P^{M^\pm}(r) \le \mathrm{C}_P^{M^\pm}(r,\lambda);$$

(iii) $K^M \le \lambda$ implies

$$\mathrm{A}_P^{M^\pm}(r) \ge \mathrm{C}_P^{M^\pm}(r,\lambda).$$

10.4 Prove the following generalizations of Theorems 10.26 and 10.27.

Theorem 10.30. *Let B be a region in a complete surface M. Suppose that every point p with* distance$(p,B) \le r$ *has a unique shortest geodesic from it to B.*

(i) *If $K^M \ge \lambda$, then*

$$\begin{aligned}\operatorname{Area}(B_r) \;\le\; & \operatorname{Area}(B) + \operatorname{Length}(\partial B)\left(\frac{\sin(r\sqrt{\lambda})}{\sqrt{\lambda}}\right) \qquad (10.62)\\ & + \pi\left\{2\chi(B) - \int_B K^M dB\right\}\left(\frac{1-\cos^3(r\sqrt{\lambda})}{3\sqrt{\lambda}}\right).\end{aligned}$$

(ii) *If $K^M \le \lambda$, then the inequality in* (10.62) *is reversed.*

Theorem 10.31. *Let B be a region in a complete* 3*-dimensional Riemannian manifold M. Denote by $k_1(\partial B)$ the integrated mean curvature of ∂B and by $K^M|\partial B$ the restriction of K^M to the plane sections tangent to ∂B.*

(i) *If $K^M \ge \lambda$, then*

$$\begin{aligned}\operatorname{Vol}(B_r) \;\le\; & \operatorname{Vol}(B) + \operatorname{Area}(\partial B)\left(\frac{r}{2} + \frac{\sin(2r\sqrt{\lambda})}{4\sqrt{\lambda}}\right) \qquad (10.63)\\ & + k_1(\partial B)\left(\frac{1-\cos^4(r\sqrt{\lambda})}{4\sqrt{\lambda}}\right)\\ & + \left\{2\pi\chi(B) - \int_{\partial B}(K^M|_{\partial B})d(\partial B)\right\}\left(\frac{r}{\sqrt{\lambda}} - \frac{\sin(2r\sqrt{\lambda})}{4\lambda\sqrt{\lambda}}\right).\end{aligned}$$

(ii) *If $K^M \le \lambda$, then the inequality in* (10.63) *is reversed.*

Chapter 11

Mean-value Theorems

This chapter is devoted to an exposition of the results of [GW] and related papers.

Recall that a function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be **harmonic** provided

$$\Delta f = \frac{\partial^2 f}{\partial u_1^2} + \cdots + \frac{\partial^2 f}{\partial u_n^2} = 0.$$

One of the most important properties of such a harmonic function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is that it has the **mean-value property**, that is,

$$\frac{1}{\text{volume}\,\big(S^{n-1}(r)\big)} \int_{S^{n-1}(r)} f(m+u)du = f(m), \tag{11.1}$$

where $S^{n-1}(r)$ denotes a sphere of radius r centered at $m \in \mathbb{R}^n$. How can this mean-value theorem be generalized? On the one hand, we can replace $\mathbb{R}^n$ by an arbitrary Riemannian manifold M. Instead of the left-hand side of (11.1) it is appropriate to use

$$M_m(r,f) = \frac{1}{\text{volume}\,\big(\exp_m(S^{n-1}(r))\big)} \int_{\exp_m(S^{n-1}(r))} f{*}d\sigma, \tag{11.2}$$

where $\exp_m$ is the exponential map and $*d\sigma$ is the volume form of $\exp_m(S^{n-1}(r))$. Note that $M_m(r,f)$ makes sense for any Riemannian manifold M and any integrable real-valued function $f: M \longrightarrow \mathbb{R}^n$, provided r is sufficiently small. Geometrically, $M_m(r,f)$ is the mean-value of f over a geodesic sphere of radius r in M.

A generalization of a different sort is Pizzetti's[1] formula (see [CH], [Pizz1] and [Pizz2]). This mean-value theorem for an analytic function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ asserts

[1]Paolo Pizzetti (1860–1918). Italian geodesist, who taught at the Universities of Genova and Pisa. His **Trattato di geodesia teoretica** was published in Bologna in 1905 and his **Principii della teoria meccanica della figura dei pianeti** was published in Genova in 1913. Although Courant and Hilbert make effective use of his formula in [CH], unfortunately they misspell his name.

that

$$M_m(r,f) = \Gamma\Big(\frac{n}{2}\Big)\sum_{k=0}^{\infty}\frac{(\Delta^k f)_m}{k!\Gamma\Big(\dfrac{n}{2}+k\Big)}\Big(\frac{r}{2}\Big)^{2k}, \tag{11.3}$$

where Δ denotes the Laplacian of $\mathbb{R}^n$. Of course, if f is harmonic, that is, if $\Delta f = 0$, then (11.3) reduces to (11.1). A slick derivation of (11.3) using the Fourier transform has been given in [Zalc].

In this section we combine the mean value formulas (11.2) and (11.3). The result is the Taylor expansion for $M_m(r,f)$ in powers of r, where f is an arbitrary analytic function on a Riemannian manifold M. The formula (11.3) can be written more compactly as a Bessel function in $\sqrt{-\Delta}$:

$$M_m(r,f) = \big(j_{(n/2)-1}(r\sqrt{-\Delta})\big)[f](m), \tag{11.4}$$

where $j_\ell(z) = 2^\ell\Gamma(\ell+1)J_\ell(z)/z^\ell$ and J_ℓ denotes a Bessel function of the first kind of order ℓ. It is remarkable that J_ℓ, usually considered a rather complicated function, arises in this natural way. The mean-value function $M_m(r,f)$ can be thought of as a generalization of a Bessel function that is associated with each point of a Riemannian manifold. Of course, for a homogeneous Riemannian manifold, the function $M_m(r,f)$ does not depend on m. If M is a rank 1 symmetric space, then $M_m(r,f)$ satisfies a differential equation that generalizes the Bessel equation.

For comparison theorems on $M_m(r,f)$ similar to those given in Chapter 8 for the volume, see [Savo].

11.1 The Laplacian and the Euclidean Laplacian

The technique to find $M_m(r,f)$ for a general Riemannian manifold M is to use the exponential map $\exp_m$ to transfer formulas between M and the Euclidean space M_m. In this way, we obtain a formula similar to (11.3), but unfortunately the right-hand side is expressed not in terms of the Laplacian Δ of the Riemannian manifold, but in terms of another operator $\tilde{\Delta}_m$, which we call the **Euclidean Laplacian**. If $(x_1,\ldots,x_m)$ is any system of normal coordinates at m, then $\tilde{\Delta}_m$ is given by

$$\tilde{\Delta}_m = \sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}. \tag{11.5}$$

This contrasts with the ordinary Laplacian of a Riemannian manifold M given by

$$\Delta = \frac{1}{\sqrt{g}}\sum_{ij=1}^{n}\frac{\partial}{\partial y_i}\left(g^{ij}\sqrt{g}\frac{\partial}{\partial y_j}\right), \tag{11.6}$$

where now $(y_1,\ldots,y_n)$ is *any* coordinate system, g_{ij} are the components of the metric tensor of M relative to $(y_1,\ldots,y_n)$, (g^{ij}) is the inverse matrix of (g_{ij}) and

$g = \det(g_{ij})$. Although $(\Delta f)_m = (\tilde{\Delta}_m f)_m$, it is in general false that $(\Delta^k f)_m = (\tilde{\Delta}^k_m f)_m$ for $k > 1$. A function $f\colon M \longrightarrow \mathbb{R}$ is called **harmonic** if $\Delta f = 0$.

Note that (11.6) is valid for any coordinate system, but that (11.5) holds only for a normal coordinate system centered at m; it is for this reason that we write $\tilde{\Delta}_m$ with a subscript. Nevertheless, (11.5) is independent of the choice of normal coordinates at m. In this section we show that there is a globally defined differential operator L on M such that L^k coincides at m with $\tilde{\Delta}^k_m$.

Let ∇^k denote the k^{th} power of the covariant derivative of M (see Section 9.1). For a local orthonormal frame $\{E_1, \ldots, E_n\}$ we write

$$\nabla^k_{E_{i_1} \ldots E_{i_k}}$$

more simply as $\nabla^k_{i_1 \cdots i_k}$. The following lemma is easy to establish using the methods of Section 9.1:

Lemma 11.1. *Let $f\colon M \longrightarrow \mathbb{R}$ be a C^∞ function, and let $X, X_1, \ldots, X_k \in \mathfrak{X}(M)$ be normal coordinate vector fields at m. Then*

(i) $\nabla^k_{X_1 \ldots X_{k-1} X_k} f = \nabla^k_{X_1 \ldots X_{k-2} X_k X_{k-1}} f$;

(ii) $\nabla^k_{X \ldots X} f = (X^k f)_m$;

(iii) $\Delta^k f = \displaystyle\sum_{i_1 \ldots i_k = 1}^{n} \nabla^{2k}_{i_1 i_1 \ldots i_k i_k} f$ *for any local orthonormal frame* $\{E_1, \ldots, E_n\}$.

Proof. For (i) we observe that

$$\nabla_{X_{k-1} X_k} f - \nabla_{X_k X_{k-1}} f = [X_{k-1}, X_k] f = 0. \tag{11.7}$$

When we apply $\nabla^{k-2}_{X_1 \ldots X_{k-2}}$ to both sides of (11.7), we get (i). Next (ii) holds because the integral curve of X through m is a geodesic on which X is parallel. Finally, (iii) is a consequence of the fact that

$$\Delta f = \sum_{i_1=1}^{n} \nabla^2_{i_1 i_1} f. \qquad \square$$

Recall that $\mathfrak{Q}_s$ is the subset of the symmetric group $\mathfrak{S}_{2s}$ defined by

$$\mathfrak{Q}_s = \Big\{ \sigma \in \mathfrak{S}_{2s} \mid \sigma_1 < \sigma_3 < \cdots < \sigma_{2s-1} \text{ and } \sigma_{2t-1} < \sigma_{2t} \text{ for } t = 1, \ldots, s \Big\}.$$

Also, let $S^{n-1}(1)$ denote the unit sphere in a tangent space M_m. As a special case of Lemma 4.5, page 61 we have the following fact:

Corollary 11.2. *Let f be a real-valued function that is analytic near $m \in M$. Then*

$$\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2}} \int_{S^{n-1}(1)} (\nabla^{2k}_{x\ldots x} f)_m dx \tag{11.8}$$
$$= \frac{1}{n(n+2)\cdots(n+2k-2)} \sum_{\sigma\in\Omega_k} \sum_{i_1\ldots i_k=1}^{n} \sigma(\nabla^{2k}_{i_1 i_1 \ldots i_k i_k} f)_m.$$

For example,

$$\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2}} \int_{S^{n-1}(1)} (\nabla^2_{xx} f)_m dx = \frac{1}{n}\sum_{i=1}^{n} (\nabla^2_{ii} f)m$$

and

$$\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2}} \int_{S^{n-1}(1)} (\nabla^4_{xxxx} f)_m dx$$
$$= \frac{1}{n(n+2)} \sum_{ij=1}^{n} (\nabla^4_{iijj} f + \nabla^4_{ijij} f + \nabla^4_{ijji} f)_m.$$

The space of normal coordinate vector fields at m is canonically isomorphic to the tangent space M_m, and so the unit spheres in each of these spaces are isometric to $S^{n-1}(1)$. Instead of integrating

$$(\nabla^{2k}_{x\ldots x} f)_m$$

over $S^{n-1}(1)$, we can integrate X^{2k}, where X is a unit-length normal coordinate vector field at m.

Corollary 11.3. *We have*

$$\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2}} \int_{S^{n-1}(1)} (X^{2k} f)_m dX \tag{11.9}$$
$$= \frac{1\cdot 3\cdots(2k-1)}{n(n+2)\cdots(n+2k-2)} \sum_{i_1\ldots i_k=1}^{n} (X^2_{i1}\cdots X^2_{ik} f)_m,$$

where f is a real-valued function analytic near $m \in M$.

Proof. The right-hand side of (11.9) is symmetric, that is, it is independent of the order of the X_i's; therefore, (11.9) follows from (4.13), page 62. $\square$

Now we can find a formula for $(\tilde{\Delta}^k_m f)_m$ in terms of ∇^{2k}.

Lemma 11.4. *We have*

$$1 \cdot 3 \cdots (2k-1)(\tilde{\Delta}^k_m f)_m = \sum_{\sigma \in \Omega_k} \sum_{i_1 \ldots i_k = 1}^{n} \sigma(\nabla^{2k}_{i_1 i_1 \ldots i_k i_k} f)_m, \tag{11.10}$$

for any real-valued function f near $m \in M$.

Proof. Let $X_i = \partial/\partial x_i$, where $(x_1, \ldots, x_n)$ is a system of normal coordinates at m. Then

$$(\tilde{\Delta}^k_m f)_m = \sum_{i_1 \ldots i_k = 1}^{n} (X^2_{i1} \cdots X^2_{ik} f)_m. \tag{11.11}$$

From (11.9) and (11.11) we get

$$\begin{aligned} &1 \cdot 3 \cdots (2k-1)(\tilde{\Delta}^k_m f)_m \\ &= \frac{n(n+2)\cdots(n+2k-2)\Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2}} \int_{S^{n-1}(1)} (X^{2k} f)_m dX. \end{aligned} \tag{11.12}$$

Now part (ii) of Lemma 11.1 implies that

$$\begin{aligned} &1 \cdot 3 \cdots (2k-1)(\tilde{\Delta}^k_m f)_m \\ &= \frac{n(n+2)\cdots(n+2k-2)\Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2}} \int_{S^{n-1}(1)} \left(\nabla^{2k}_{X \ldots X}\right)_m dX. \end{aligned} \tag{11.13}$$

Then (11.10) follows from (11.8) and (11.13). □

Corollary 11.5. $(\tilde{\Delta}^k_m f)_m = (L^k f)_m$, *where L^k is a globally defined differential operator of degree $2k$ on M.*

Proof. We define L^k by

$$L^k f = \frac{1}{1 \cdot 3 \cdots (2k-1)} \sum_{i_1 \ldots i_k}^{n} \left(\nabla_{i_1 i_1 \ldots i_k i_k} f + \cdots + \nabla_{i_1 \ldots i_k i_k \ldots i_1} f\right). \tag{11.14}$$

Since ∇^{2k} is a tensor field, the right-hand side of (11.14) does not depend on the choice of orthonormal frame field. □

11.2 Relations between the Two Laplacians and Curvature

Next we turn to the calculation of the powers of $\tilde{\Delta}_m$ in terms of Δ and the curvature of M. We see from (11.10) that it suffices to compute the various permuted powers ∇^{2k}, that is, the expressions of the form

$$\sum_{\sigma\in\Omega_k}\sum_{i_1\ldots i_k=1}^{n}\sigma(\nabla^{2k}_{i_1 i_1\ldots i_k i_k}f).$$

This can be done in principle for any k using various curvature identities. However, as k becomes large the calculations quickly become very complicated, even though part (i) of Lemma 11.1 cuts the work in half. We do the calculations for $k \leq 2$, leaving $k = 3$ for the exercises. Of course, it is obvious from (11.10) that

Lemma 11.6. *We have*

$$(\tilde{\Delta}_m f)_m = (\Delta f)_m = \left(\sum_{i=1}^{n}\nabla^2_{ii}f\right)_m$$

for any real-valued function f near $m \in M$.

Next, let R_{ijkl} and ρ_{ij} denote the components of the curvature and Ricci tensors with respect to an orthonormal basis of M_m, and let τ denote the scalar curvature. For the computation of the permuted second powers of the Laplacian (as well as $\tilde{\Delta}^2_m$) we introduce the following invariants:

$$\langle df, d\tau\rangle = \langle\nabla f, \nabla\tau\rangle = \sum_{i=1}^{n}(\nabla_i f)(\nabla_i\tau), \tag{11.15}$$

and

$$\left\langle\nabla^2 f, \rho\right\rangle = \sum_{ij=1}^{n}(\nabla^2_{ij}f)\rho_{ij}. \tag{11.16}$$

(Note that $(\nabla^2 f)_m$ is just the Hessian of f at m.) These are the analogs of the linear curvature invariants given by Lemma 4.4, page 59.

The permuted second powers of the Laplacian (other than ∇^2) are

$$\sum_{ij=1}^{n}\nabla^4_{ijij} \qquad \text{and} \qquad \sum_{ij=1}^{n}\nabla^4_{ijji}.$$

They are expressed in terms of Δ^2 and the invariants (11.15) and (11.16) as follows.

Lemma 11.7. *Let f be an analytic function defined on an open subset of M. Then we have*

$$\sum_{ij=1}^{n} \nabla^4_{ijij} f = \sum_{ij=1}^{n} \nabla^4_{ijji} f = \Delta^2 f + \frac{1}{2} \langle \nabla f, \nabla \tau \rangle + \langle \nabla^2 f, \rho \rangle . \tag{11.17}$$

Proof. It is clear from part (i) of Lemma 11.1 that $\sum \nabla^4_{ijij}$ and $\sum \nabla^4_{ijji}$ are the same. Moreover, using the Ricci identity (9.25), page 194, it is easy to see that

$$\sum_{j=1}^{n} (\nabla^3_{jij} f - \nabla^3_{ijj} f) = \sum_{k=1}^{n} \rho_{ik} \nabla_k f \tag{11.18}$$

for $i = 1, \ldots, n$. When we apply ∇_i to both sides of (11.18) and sum over i, we get

$$\sum_{ij=1}^{n} (\nabla^4_{ijij} f - \nabla^4_{iijj} f) = \sum_{ik=1}^{n} (\nabla_i \rho_{ik} \nabla_k f + \rho_{ik} \nabla^2_{ik} f). \tag{11.19}$$

Then (11.17) follows from (11.19) and (9.27), page 195. □

11.3 The Power Expansion for the Mean-Value $M_m(r, f)$

As in Section 9.1 we let $(x_1, \ldots, x_n)$ be a system of normal coordinates at a point m in a Riemannian manifold M, and we put $X_i = \partial/\partial x_i$ for $i = 1, \ldots, n$. Then $X_1, \ldots, X_n$ are normal coordinate vector fields that are orthonormal at m. Also, we write

$$\theta = \omega_{1 \ldots n} = \omega(X_1, \ldots, X_n),$$

where ω is a Riemannian volume element of M. First, we note the following fact.

Lemma 11.8. *Let f be a real-valued function possessing at least two derivatives defined in a neighborhood of m. Let $\bar{\Delta}$ be the Laplacian of the Euclidean space M_m. Then*

$$(\tilde{\Delta}_m f) \circ \exp_m = \bar{\Delta}(f \circ \exp_m).$$

Proof. This follows from the fact that the normal coordinate vector fields $\partial/\partial x_i$ are just the images under $\exp_m$ of the natural coordinate vector fields on M_m. □

Now we determine the complete expansion for $M_m(r, f)$ in terms of $\tilde{\Delta}_m$.

Lemma 11.9. *We have*

$$\int_{\exp_m(S^{n-1}(r))} f * d\sigma = 2\pi^{n/2} r^{n-1} \sum_{k=0}^{\infty} \left(\frac{r}{2}\right)^{2k} \frac{1}{k! \Gamma\left(\frac{n}{2} + k\right)} \tilde{\Delta}^k_m (f\theta)(m), \tag{11.20}$$

where f is any real-valued function analytic near $m \in M$.

Proof. To compute the right-hand side of (11.20), we must first change variables so that we can carry out the integration over an ordinary sphere in M_m. This change of variables is carried out just as in Lemma 3.12, page 41; exactly the same technique yields

$$\int_{\exp_m(S^{n-1}(r))} f{*}d\sigma = \int_{S^{n-1}(1))} (f\theta)\left(\exp_m(r\,u)\right) du. \tag{11.21}$$

(For the tube generalization of (11.21), see Problem 3.13, page 52.)

The right-hand side of (11.21), being an integral over an ordinary sphere, can be computed using Pizzetti's formula (11.3), which we rewrite as

$$\int_{S^{n-1}(r)} h\,du = 2\pi^{n/2} r^{n-1} \sum_{k=0}^{\infty} \frac{(\bar{\Delta}^k h)_m}{k!\Gamma\left(\dfrac{n}{2}+k\right)} \left(\frac{r}{2}\right)^{2k}, \tag{11.22}$$

where h is a real-valued function on $\mathbb{R}^n$. Then (11.20) follows from (11.21) and (11.22). $\square$

Taking $\theta = 1$ in (11.20), we obtain:

Corollary 11.10. *The volume of a geodesic sphere* $\exp_m(S^{n-1}(r))$ *in a Riemannian manifold* M *is given by the formula*

$$\text{volume}(\exp_m(S^{n-1}(r))) = 2\pi^{n/2} r^{n-1} \sum_{k=0}^{\infty} \frac{\tilde{\Delta}_m^k(\theta)_m}{k!\left(\dfrac{n}{2}+k\right)} \left(\frac{r}{2}\right)^{2k}.$$

Now we can write down the formula for $M_m(r,f)$ in terms of $\tilde{\Delta}_m$.

Theorem 11.11. *Let* f *be a real-valued function on a Riemannian manifold* M *that is analytic near* $m \in M$. *Then the mean-value of* f *over* $\exp_m(S^{n-1}(r))$ *is given by*

$$\begin{aligned} M_m(r,f) &= \frac{\left(\jmath_{(n/2)-1}(r\sqrt{-\tilde{\Delta}_m})\right)[f\theta]_m}{\left(\jmath_{(n/2)-1}(r\sqrt{-\tilde{\Delta}_m})\right)[\theta]_m} \\ &= \frac{\displaystyle\sum_{k=0}^{\infty} \frac{\tilde{\Delta}_m^k(f\theta)_m}{k!\Gamma\left(\dfrac{n}{2}+k\right)} \left(\frac{r}{2}\right)^{2k}}{\displaystyle\sum_{k=0}^{\infty} \frac{\tilde{\Delta}_m^k(\theta)_m}{k!\Gamma\left(\dfrac{n}{2}+k\right)} \left(\frac{r}{2}\right)^{2k}}. \end{aligned} \tag{11.23}$$

Proof. Equation (11.23) follows immediately from Lemma (11.9) and Corollary 11.10. $\square$

In principle, we know that $\tilde{\Delta}_m$ can be expressed in terms of Δ and the curvature of M. By dividing the two power series on the right-hand side of (11.23) it would then be possible to compute the first few terms of the power series expansion for $M_m(r, f)$. In practice this method is very tedious, so we give an alternate method.

Theorem 11.12. *Let $f\colon M \longrightarrow \mathbb{R}$ be an analytic real-valued function on an analytic Riemannian manifold M, and let $m \in M$. Then*

$$M_m(r, f) = f(m) + A(n)_m r^2 + B(n)_m r^4 + C(n)_m r^6 + O(r^8) \tag{11.24}$$

as $r \longrightarrow 0$, where

$$\begin{cases} A(n) &= \dfrac{\Delta f}{2n}, \\ B(n) &= \dfrac{3\Delta^2 f - 2\left\langle \nabla^2 f, \rho \right\rangle - 3\left\langle \nabla f, \nabla \tau \right\rangle + \dfrac{4\tau\, \Delta f}{n}}{24n(n+2)}. \end{cases}$$

(For $C(n)$ see Problem 11.5, page 245.)

Proof. The general form of the power series of a function f in terms of the covariant derivative is clear:

$$\begin{aligned} f &= f(m) + \sum_{i=1}^{n} (\nabla_i f)_m x_i + \frac{1}{2} \sum_{ij=1}^{n} (\nabla^2_{ij} f)_m x_i x_j \\ &\quad + \frac{1}{6} \sum_{ijk=1}^{n} (\nabla^3_{ijk} f)_m x_i x_j x_k + \frac{1}{24} \sum_{ijk\ell=1}^{n} (\nabla^4_{ijk\ell} f)_m x_i x_j x_k x_\ell + \cdots . \end{aligned} \tag{11.25}$$

(In fact, (11.25) is a special case of (9.18), page 191.) When we multiply the power expansion for θ (formula (9.22), page 193 by (11.25) we obtain:

$$\begin{aligned} f\theta &= f(m) + \sum_{i=1}^{n} (\nabla_i f)_m x_i + \frac{1}{6} \sum_{ij=1}^{n} (3\nabla^2_{ij} f - f\rho_{ij})_m x_i x_j \\ &\quad + \frac{1}{12} \sum_{ijk=1}^{n} (2\nabla^3_{ijk} f - 2(\nabla_i f)\rho_{jk} - f\nabla_i \rho_{jk})_m x_i x_j x_k \\ &\quad + \frac{1}{24} \sum_{ijk\ell=1}^{n} \Bigg(\nabla^4_{ijk\ell} f - 2(\nabla^2_{ij} f)\rho_{k\ell} - 2(\nabla_i f)(\nabla_j \rho_{k\ell}) \\ &\quad + f\Bigg(-\frac{3}{5}\nabla^2_{ij}\rho_{k\ell} + \frac{1}{3}\rho_{ij}\rho_{k\ell} - \frac{2}{15} \sum_{ab=1}^{n} R_{iajb} R_{ka\ell b} \Bigg)\Bigg)_m x_i x_j x_k x_\ell + \cdots . \end{aligned} \tag{11.26}$$

Next, we integrate (11.26) over $\exp_m(S^{n-1}(r))$ term using the moment formulas from Section A.2 of the Appendix. The computations generalize those of the integrals of the μ in Section 9.2. Just as in Section 9.2, integrals of terms involving

an odd number of x_i vanish. Furthermore,

$$\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2}}\int_{\exp_m(S^{n-1}(r))}\sum_{ij=1}^{n}(3\nabla^2_{ij}f - f\rho_{ij})_m x_i x_j dA \tag{11.27}$$
$$= \frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2}}\sum_{i=1}^{n}(3\nabla^2_{ii}f - f\rho_{ii})_m r^2\int_{S^{n-1}(1)} a_i^2 du = \frac{1}{n}(3\Delta f - f\tau)_m,$$

and

$$\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2}}\int_{\exp_m(S^{n-1}(r))}\sum_{ijk\ell=1}^{n}\Big(\nabla^4_{ijk\ell}f - 2(\nabla^2_{ij}f)\rho_{k\ell} \tag{11.28}$$
$$-2(\nabla_i f)(\nabla_j\rho_{k\ell})\Big)_m x_i x_j x_k x_\ell\, dA$$
$$= \frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2}}\sum_{ij=1}^{n}\Big(\nabla^4_{iijj}f + \nabla^4_{ijij}f + \nabla^4_{ijji} - 2(\nabla^2_{ii}f)\rho_{jj} - 4(\nabla^2_{ij}f)\rho_{ij}$$
$$-2(\nabla_i f)(\nabla_i\rho_{jj}) - 4(\nabla_i f)(\nabla_j\rho_{ij})\Big)_m r^4\int_{S^{n-1}(1)} a_i^2 a_j^2 du$$
$$= \frac{1}{n(n+2)}(3\Delta^2 f - 3\langle\nabla f, \nabla r\rangle - 2\langle\nabla^2 f, \rho\rangle - 2r\Delta f)_m.$$

Therefore, we have

$$\frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{n/2}}\int_{\exp_m(S^{n-1}(r))} f*d\sigma = f(m) - \frac{(3\Delta f - f\tau)_m}{6n}r^2 \tag{11.29}$$
$$+\frac{1}{24n(n+2)}\Bigg(3\Delta^2 f - 2\langle\nabla^2 f, \rho\rangle - 3\langle\nabla f, \nabla\tau\rangle - 2\tau\Delta f$$
$$+f\left(-\frac{6}{5}\Delta\tau + \frac{1}{3}\tau^2 + \frac{8}{15}\|\rho\|^2 - \frac{1}{5}\|R\|^2\right)\Bigg)_m r^4 + O(r^6).$$

In particular, taking $f = 1$ we obtain

$$\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2}}\int_{\exp_m(S^{n-1}(r))} *d\sigma = 1 - \frac{\tau_m}{6n}r^2 \tag{11.30}$$
$$+\frac{1}{24n(n+2)}\left(-\frac{6}{5}\Delta\tau + \frac{1}{3}\tau^2 + \frac{8}{15}\|\rho\|^2 - \frac{1}{5}\|R\|^2\right)_m r^4 + O(r^6).$$

(Formula (11.30) can also be obtained by taking the derivatives of both sides of (9.30), page 196.) When we divide (11.29) by (11.30), we obtain (11.24). $\square$

A classical result of Willmore [Wil] is a characterization of harmonic spaces by means of the mean-value property for harmonic functions. More precisely:

Theorem 11.13. *Let M be a Riemannian manifold. Then m is a harmonic space if and only if every harmonic function on M has the mean-value property*

$$M_m(r, f) = f(m) \tag{11.31}$$

for all small r.

Since every rank 1 symmetric space is a harmonic space, this result implies that (11.31) holds for spheres, complex projective spaces, quaternionic projective spaces, the Cayley plane and their duals.

We are now in a position to generalize Willmore's Theorem. We next show that other classes of Riemannian manifolds satisfy mean-value properties that are weaker than (11.31).

Theorem 11.14. *Let M be an analytic Riemann manifold, and let $m \in M$.*

(i) *Any function f harmonic near m satisfies*

$$M_m(r, f) = f(m) + O(r^4) \qquad \text{as } r \longrightarrow 0. \tag{11.32}$$

(ii) *If M is an Einstein manifold and f harmonic near m, then*

$$M_m(r, f) = f(m) + O(r^6) \qquad \text{as } r \longrightarrow 0. \tag{11.33}$$

(iii) *If M is an irreducible symmetric space and f harmonic near m, then*

$$M_m(r, f) = f(m) + O(r^8) \qquad \text{as } r \longrightarrow 0. \tag{11.34}$$

Proof. If $\Delta f = 0$, then $A(n) = 0$ in (11.24), proving (i). Furthermore, $B(n)$ reduces to

$$B(n) = \frac{-2\langle \nabla^2 f, \rho\rangle - 3\langle \nabla f, \nabla \tau\rangle}{24n(n+2)}. \tag{11.35}$$

If M is an Einstein manifold, then the scalar curvature of M is constant, and so the second term on the right-hand side of (11.35) vanishes. Moreover, the Ricci curvature of M satisfies $\rho(X, Y) = \lambda\langle X, Y\rangle$, for all vector fields X and Y (where λ is a constant), and so

$$\langle \nabla^2 f, \rho\rangle = \sum_{ij=1}^{n} (\nabla^2_{ij} f)(\rho_{ij}) = (\Delta f)\frac{\tau}{n} = 0,$$

so that the first term vanishes as well. This proves (ii).

Finally, to prove (iii), we observe that if M is an Einstein manifold and $\Delta f = 0$, then (11.45) (in Problem 11.5) reduces to

$$C(n) = -\frac{\langle \nabla^2 f \dot{R}\rangle}{90n(n+2)(n+4)}. \tag{11.36}$$

In particular, (11.36) holds for an irreducible symmetric space.

Consider the tensor field $\dot{R}$ defined by

$$\dot{R}(X,Y) = \sum_{abc=1}^{n} R_{abcX}R_{abcY} \tag{11.37}$$

for $X, Y \in \mathfrak{X}(M)$; note that $\dot{R}$ is symmetric. If M is an irreducible symmetric space, then $\dot{R}$, a scalar multiple of the metric tensor, that is,

$$\dot{R}(X,Y) = \mu\langle X,Y\rangle \tag{11.38}$$

for all $X, Y \in \mathfrak{X}(M)$ for some constant μ. An easy calculation using (11.37) and (11.38) shows that (11.45) reduces to $C(n) = 0$. □

Any simple Lie group G is an irreducible symmetric space, so (11.34) holds for G. In [CGW] it is shown that a harmonic function on the exceptional Lie group E^8 satisfies

$$M_m(r,f) = f(m) + O(r^{10}).$$

11.4 Problems

11.1 Show that

$$\sum_{j=1}^{n} \nabla^4_{ijjk} f = \nabla^2_{ik}(\Delta f) + \sum_{\ell=1}^{n}(\nabla_i\rho_{k\ell}\nabla_\ell f + \rho_{k\ell}\nabla^2_{i\ell} f), \tag{11.39}$$

$$\begin{aligned}\sum_{i=1}^{n} \nabla^4_{ijik} f &= \nabla^2_{jk}(\Delta f) + \sum_{\ell=1}^{n}(\nabla_j\rho_{k\ell}\nabla_\ell f + \rho_{k\ell}\nabla^2_{j\ell} f + \rho_{j\ell}\nabla^2_{k\ell} f)\\ &\quad + \sum_{il=1}^{n} R_{ijk\ell}\nabla^2_{i\ell} f,\end{aligned} \tag{11.40}$$

and

$$\begin{aligned}\sum_{i=1}^{n} \nabla^4_{iijk} f &= \nabla^2_{jk}(\Delta f) + \sum_{\ell=1}^{n}((\nabla_j\rho_{k\ell} + \nabla_k\rho_{j\ell} - \nabla_\ell\rho_{jk})\nabla_\ell f\\ &\quad + \rho_{k\ell}\nabla^2_{j\ell} f + \rho_{j\ell}\nabla^2_{k\ell} f) + 2\sum_{il=1}^{n} R_{ijk\ell}\nabla^2_{i\ell} f.\end{aligned} \tag{11.41}$$

11.2 The computation of $\tilde{\Delta}^3 f$ is complicated. We define

$$\langle\nabla^2(\Delta f), \rho\rangle = \sum_{ijk=1}^{n} \rho_{ij}\nabla^4_{ijkk} f,$$

$$\langle \nabla(\Delta f), \nabla \tau \rangle = \sum_{ij=1}^{n} (\nabla \tau_i)(\nabla^3_{ijj} f),$$

$$\left\langle \nabla^3 f, \nabla \rho \right\rangle = \sum_{ijk=1}^{n} (\nabla_i \rho_{jk})(\nabla^3_{ijk} f),$$

$$\left\langle \nabla^2 f, \nabla^2 \tau \right\rangle = \sum_{ij=1}^{n} (\nabla^2_{ij} \tau)(\nabla^2_{ij} f),$$

$$\left\langle \nabla^2 f, \Delta \rho \right\rangle = \sum_{ijk=1}^{n} (\nabla^2_{ii} \rho_{jk})(\nabla^2_{jk} f),$$

$$\left\langle \nabla^2 f \otimes \rho, \bar{R} \right\rangle = \sum_{ijk\ell=1}^{n} R_{ikj\ell} \rho_{ij} (\nabla^2_{kl} f),$$

$$\sigma(\nabla^2 f \otimes \rho \otimes \rho) = \sum_{ijk=1}^{n} \rho_{ij} \rho_{jk} (\nabla^2_{ki} f),$$

$$\left\langle \nabla^2 f, \dot{R} \right\rangle = \sum_{ijk\ell h=1}^{n} R_{ijk\ell} R_{ijkh} (\nabla^2_{ih} f),$$

$$\langle \nabla f, \nabla(\Delta \tau) \rangle = \sum_{ij=1}^{n} (\nabla^3_{ijj} \tau)(\nabla_i f),$$

$$\rho(\nabla f, \nabla \tau) = \sum_{ij=1}^{n} \rho_{ij} (\nabla_i \tau)(\nabla_i f),$$

$$\langle \rho \otimes \nabla f, \nabla \rho \rangle = \sum_{ijk=1}^{n} (\nabla_i \rho_{jk}) \rho_{ij} (\nabla_k f),$$

$$\langle \nabla f \otimes \rho, \nabla \rho \rangle = \sum_{ijk=1}^{n} (\nabla_i \rho_{jk})(\nabla_i f) \rho_{jk},$$

$$\left\langle \nabla f \otimes \nabla \rho, \bar{R} \right\rangle = \sum_{ijk\ell=1}^{n} (\nabla_i f)(\nabla_j \rho_{kl}) R_{ikjl},$$

$$\langle \nabla f \otimes R, \nabla R \rangle = \sum_{ijk\ell h=1}^{n} (\nabla_i f) R_{jk\ell h} (\nabla_i R_{jk\ell h}).$$

Contract (11.39) to obtain

$$\sum_{ijk=1}^{n} \rho_{jk}\nabla^4_{jiik} f = \langle \nabla^2(\Delta f), \rho\rangle + \sigma(\nabla^2 f \otimes \rho \otimes \rho) + \langle \rho \otimes \nabla f, \nabla\rho\rangle. \quad (11.42)$$

Contract (11.40) to obtain

$$\begin{aligned}\sum_{i=1}^{n} \rho_{jk}\nabla^4_{ijik} f &= \langle \nabla^2(\Delta f), \rho\rangle + 2\sigma(\nabla^2 f \otimes \rho \otimes \rho) \quad (11.43)\\ &\quad +\langle \rho \otimes \nabla f, \nabla\rho\rangle - \langle \nabla^2 f \otimes \rho, \bar{R}\rangle.\end{aligned}$$

Contract (11.41) to obtain

$$\begin{aligned}\sum_{i=1}^{n} \rho_{jk}\nabla^4_{iijk} f &= \langle \nabla^2(\Delta f), \rho\rangle + 2\sigma(\nabla^2 f \otimes \rho \otimes \rho) \quad (11.44)\\ &\quad +2\langle \rho \otimes \nabla f, \nabla\rho\rangle - 2\langle \nabla^2 f \otimes \rho, \bar{R}\rangle - \langle \nabla f \otimes \rho, \nabla\rho\rangle.\end{aligned}$$

11.3 Show that

$$\sum_{ijk=1}^{n} \nabla^6_{ijijkk} f = \sum_{ijk=1}^{n} \nabla^6_{ijjikk} = \Delta^3 f + \langle \nabla^2(\Delta f), \rho\rangle + \frac{1}{2}\langle \nabla(\Delta f), \nabla\tau\rangle,$$

$$\begin{aligned}\sum_{ijk=1}^{n} \nabla^6_{iijkjk} f &= \Delta^3 f + \langle \nabla^2(\Delta f), \rho\rangle + \frac{1}{2}\langle \nabla(\Delta f), \nabla\tau\rangle + 2\langle \nabla^3 f, \nabla\rho\rangle\\ &\quad + \langle \nabla^2 f, \nabla^2\tau\rangle + \langle \nabla^2 f, \Delta\rho\rangle - 2\langle \nabla^2 f \otimes \rho, \bar{R}\rangle + 2\sigma(\nabla^2 f \otimes \rho \otimes \rho)\\ &\quad + \frac{1}{2}\langle \nabla f, \nabla(\Delta\tau)\rangle + \rho(\nabla f, \nabla\tau) + 2\langle \rho \otimes \nabla f, \nabla\rho\rangle - \langle \nabla f \otimes \rho, \nabla\rho\rangle.\end{aligned}$$

$$\begin{aligned}\sum_{ijk=1}^{n} \nabla^6_{ijikjk} f &= \sum_{ijk=1}^{n} \nabla^6_{ijjkik} f\\ &= \sum_{ijk=1}^{n} \nabla^6_{iijkjk} f + \langle \nabla^2(\Delta f), \rho\rangle + \frac{1}{2}\langle \nabla(\Delta f), \nabla\tau\rangle\\ &\quad + \sigma(\nabla^2 f \otimes \rho \otimes \rho) + \frac{1}{2}\rho(\nabla f, \nabla\tau) + \langle \rho \otimes \nabla f, \nabla\rho\rangle.\end{aligned}$$

$$\sum_{ijk=1}^{n} \nabla^6_{ijkijk} f = \sum_{ijk=1}^{n} \nabla^6_{ijkjik} f$$
$$= \Delta^3 f + 3\langle \nabla^2(\Delta f), \rho\rangle + \frac{3}{2}\langle \nabla(\Delta f), \nabla\tau\rangle + 3\langle \nabla^3 f, \nabla\rho\rangle$$
$$+ 2\langle \nabla^2 f, \nabla^2\tau\rangle + 6\sigma(\nabla^2 f \otimes \rho \otimes \rho) - 4\langle \nabla^2 f \otimes \rho, \bar{R}\rangle$$
$$+ \frac{1}{2}\langle \nabla^2 f, \dot{R}\rangle + \frac{1}{2}\langle \nabla f, \nabla(\Delta\tau)\rangle + 2\rho(\nabla f, \nabla\tau) + 4\langle \rho \otimes \nabla f, \nabla\rho\rangle$$
$$- \langle \nabla f \otimes \rho, \nabla\rho\rangle + 3\langle \nabla f \otimes \nabla\rho, \bar{R}\rangle + \frac{1}{4}\langle \nabla f \otimes R, \nabla R\rangle,$$

$$\sum_{ijk=1}^{n} \nabla^6_{ijkkji} f = \sum_{ijk=1}^{n} \nabla^6_{ijkijk} f + \frac{1}{2}\langle \nabla^2, \dot{R}\rangle + \langle \nabla f \otimes \nabla\rho, \bar{R}\rangle$$
$$+ \frac{1}{4}\langle \nabla f \otimes R, \nabla R\rangle.$$

11.4 Show that

$$\begin{aligned}
\tilde{\Delta}^3 f \quad = \quad & \Delta^3 f + 2\langle \nabla^2(\Delta f), \rho\rangle + \langle \nabla(\Delta f), \nabla\tau\rangle + 2\langle \nabla^3 f, \nabla\rho\rangle \\
& + \frac{6}{5}\langle \nabla^2 f, \nabla^2\tau\rangle + \frac{2}{5}\langle \nabla^2 f, \Delta\rho\rangle - \frac{12}{5}\langle \nabla^2 f \otimes \rho, \bar{R}\rangle \\
& + \frac{52}{15}\sigma(\nabla^2 f \otimes \rho \otimes \rho) + \frac{4}{15}\langle \nabla^2 f, \dot{R}\rangle + \frac{2}{5}\langle \nabla f, \nabla(\Delta\tau)\rangle \\
& + \frac{4}{3}\rho(\nabla f, \nabla\tau) + \frac{8}{3}\langle \rho \otimes \nabla f, \nabla\rho\rangle - \frac{4}{5}\langle \nabla f \otimes \rho, \nabla\rho\rangle \\
& + \frac{4}{3}\langle \nabla f \otimes \nabla\rho, \bar{R}\rangle + \frac{2}{15}\langle \nabla f \otimes R, \nabla R\rangle\,.
\end{aligned}$$

11.5 Show that $C(n)$ in (11.24) is given by

$$\begin{aligned}
C(n) \quad = \quad & \frac{1}{720n(n+2)(n+4)}\Bigg(15\Delta^3 f - 30\langle \nabla^2(\Delta f), \rho\rangle \qquad (11.45) \\
& -45\langle \nabla(\Delta f), \nabla\tau\rangle - 30\langle {}^3 f, \nabla\rho\rangle - 36\langle \nabla^2 f, \nabla^2\tau\rangle - 12\langle \nabla^2 f, \Delta\rho\rangle \\
& +32\langle \nabla^2 f \otimes \rho, \bar{R}\rangle - 24\sigma(\nabla^2 f \otimes \rho \otimes \rho) - 8\langle \nabla^2 f, \dot{R}\rangle \\
& -30\langle \nabla f, \nabla(\Delta\tau)\rangle - 20\rho(\nabla f, \nabla\tau) - 20\langle \rho \otimes \nabla f, \nabla\rho\rangle \\
& +30\langle \nabla f \otimes \rho, \nabla\rho\rangle - 20\langle \nabla f \otimes \nabla\rho, \bar{R}\rangle - 10\langle \nabla f \otimes R, \nabla R\rangle \\
& +\frac{20}{n}(3\Delta^2 f - 2\langle \nabla^2 f, \rho\rangle - 3\langle \nabla f, \nabla\tau\rangle)\tau \\
& +\frac{4}{n}(18\Delta\tau - 18\|\rho\|^2 + 3\|R\|^2)\Delta f + \frac{80}{n^2}\tau^2\Delta f \Bigg).
\end{aligned}$$

Appendix A

We present three short expositions of some elementary facts that, as Weyl would say, every calculus student should know.

A.1 The Volume of a Ball in $\mathbb{R}^n$

We give the derivation of the formula for the volume of a geodesic ball in $\mathbb{R}^n$ using the technique described in Weyl's paper [Weyl1]. First, recall the definition of the gamma function:

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt.$$

In particular, the change of variables $t = u^2$ yields

$$\Gamma\left(\frac{\beta+1}{2}\right) = \int_{-\infty}^{\infty} e^{-u^2} |u|^\beta \, du.$$

Lemma 1.1. *The volume of the sphere of unit radius $S^{n-1}(1)$ in $\mathbb{R}^n$ is given by*

$$\text{volume}(S^{n-1}(1)) = \frac{2\Gamma(\frac{1}{2})^n}{\Gamma(\frac{n}{2})}.$$

Proof. We have

$$\Gamma\left(\frac{1}{2}\right)^n = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\left(- x_1^2 - \cdots - x_n^2 \right) dx_1 \ldots dx_n.$$

In this formula we change to polar coordinates in n dimensions. It is possible explicitly to write out these coordinates in terms of rectangular coordinates, but it is very complicated to do so. It is better to use only the information needed, which is:

$$r^2 = x_1^2 + \cdots + x_n^2 \tag{A.1}$$

and

$$dx_1 \wedge \ldots \wedge dx_n = r^{n-1} \, dA \wedge \, dr, \tag{A.2}$$

where dA is the volume form of $S^{n-1}(1)$. Then using (A.1) and (A.2), we compute

$$\begin{aligned}\Gamma\left(\frac{1}{2}\right)^n &= \int_0^\infty \int_{S^{n-1}(1)} e^{-r^2} r^{n-1} dA\, dr \\ &= \text{volume}(S^{n-1}(1)) \int_0^\infty e^{-r^2} r^{n-1} dr \\ &= \text{volume}(S^{n-1}(1)) \frac{1}{2} \int_0^\infty e^{-u} u^{n/2-1} du \\ &= \text{volume}(S^{n-1}(1)) \frac{1}{2} \Gamma\left(\frac{n}{2}\right).\end{aligned}$$

Hence the lemma follows. □

Corollary 1.2. $\Gamma(\frac{1}{2}) = (-\frac{1}{2})! = \sqrt{\pi}$.

Proof. If we take $n = 2$ in Lemma 1.1, we get

$$\Gamma\left(\frac{1}{2}\right)^2 = \text{volume}(S^1(1)) \frac{\Gamma(1)}{2} = \pi.$$ □

Combining Lemma 1.1 and Corollary 1.2, we obtain

Corollary 1.3. $\text{volume}(S^{n-1}(1)) = \dfrac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$.

Armed with these elementary facts about the gamma function, we can now give the formula for the volume of a ball in $\mathbb{R}^n$.

Lemma 1.4. *For all $r \geq 0$ we have*

$$V_m^{\mathbb{R}^n}(r) = \frac{(\pi r^2)^{n/2}}{(\frac{n}{2})!}.$$

Proof. Without loss of generality, we can take m to be the origin of $\mathbb{R}^n$. Then we have

$$\begin{aligned}V_m^{\mathbb{R}^n}(r) &= \int_0^r \text{volume}(S^{n-1}(t))\, dt \\ &= \int_0^r \frac{2\pi^{n/2} t^{n-1}}{\Gamma(\frac{n}{2})} dt \\ &= \frac{2\pi^{n/2} r^n}{n\Gamma(\frac{n}{2})} = \frac{\pi^{n/2} r^n}{(\frac{n}{2})!}.\end{aligned}$$ □

The values of $V_m^{\mathbb{R}^n}(r)$ for some small values of n are given in the following table.

n	0	1	2	3	4	5	6	7	8
$\mathrm{V}_m^{\mathbb{R}^n}(r)$	1	$2r$	πr^2	$\dfrac{4\pi r^3}{3}$	$\dfrac{\pi^2 r^4}{2}$	$\dfrac{8\pi^2 r^5}{15}$	$\dfrac{\pi^3 r^6}{6}$	$\dfrac{16\pi^3 r^7}{105}$	$\dfrac{\pi^4 r^8}{24}$

Note also the general formulas

$$\mathrm{V}_m^{\mathbb{R}^{2n}}(r) = \frac{\pi^n r^{2n}}{n!} \qquad \text{and} \qquad \mathrm{V}_m^{\mathbb{R}^{2n+1}}(r) = \frac{2^{n+1}\pi^n r^{2n+1}}{1\cdot 3\cdots(2n+1)}.$$

The following *Mathematica* function can be used to compute $\mathrm{V}_m^{\mathbb{R}^n}(r)$:

```
volsphere[n_] [r_] := PowerExpand[(Pi r^2)^(n/2)/(n/2)!]
```

Then the above table can be generated with

```
Table[volsphere[n] [r],{n,0,8}]
```

A.2 Moments

There is an interesting generalization of Corollary 1.3. For any integrable function $F\colon\mathbb{R}^n \longrightarrow \mathbb{R}$ let $\langle F\rangle$, denote the average of F over the unit sphere $S^{n-1}(1) \subset \mathbb{R}^n$:

$$\langle F\rangle = \frac{\displaystyle\int_{S^{n-1}(1)} F\,du}{\displaystyle\int_{S^{n-1}(1)} du}.$$

Let $\{e_1,\dots,e_n\}$ be an orthonormal basis of $\mathbb{R}^n$. It is easy to prove that the dual basis $\{u_1,\dots,u_n\}$ forms a global coordinate system on $\mathbb{R}^n$, and, in fact, they are normal coordinates. An important question is to compute the mean values of the polynomials $u_1^{i_1}\cdots u_n^{i_n}$. These are called **moments** (see [ST]). It is not hard to see that

$$\left\langle u_1^{i_1}\cdots u_n^{i_n}\right\rangle = 0$$

if any or all of the exponents i_j are odd; this is because the values of $u_1^{i_1}\cdots u_n^{i_n}$ taken on one hemisphere are exactly canceled by the values on the antipodal

hemisphere. Moreover, it is easy to calculate $\langle u_i^2 \rangle$ for any i. In the first place, it is clear that $\langle u_i^2 \rangle = \langle u_j^2 \rangle$ for all i and j. So

$$\langle u_i^2 \rangle = \frac{1}{n} \sum_{j=1}^{n} \langle u_j^2 \rangle = \frac{1}{n} \langle 1 \rangle = \frac{1}{n}.$$

A more interesting problem is to compute $\langle u_i^4 \rangle$. This time we have

$$\begin{aligned} 1 &= \left\langle \left(\sum_{j=1}^{n} u_j^2 \right)^2 \right\rangle = \sum_{j=1}^{n} \langle u_j^4 \rangle + 2 \sum_{1 \le i < j \le n} \langle u_i^2 u_j^2 \rangle \qquad (A.3) \\ &= n \langle u_i^4 \rangle + n(n-1) \langle u_i^2 u_j^2 \rangle, \end{aligned}$$

where $i \neq j$. To obtain a second relation between $\langle u_i^4 \rangle$ and $\langle u_i^2 u_j^2 \rangle$, we must change bases. (All of these calculations of moments are clearly independent of the choice of orthonormal basis.) So let $\{e_1', \ldots, e_n'\}$ be defined by

$$\begin{aligned} e_1' &= \frac{e_1 + e_2}{\sqrt{2}} \\ e_2' &= \frac{e_1 - e_2}{\sqrt{2}} \\ e_j' &= e_j \qquad\qquad (3 \le j \le n). \end{aligned}$$

Then the corresponding dual basis $\{u_1', \ldots, u_n'\}$ is given by

$$\begin{aligned} u_1' &= \frac{u_1 + u_2}{\sqrt{2}} \\ u_2' &= \frac{u_1 - u_2}{\sqrt{2}} \\ u_j' &= u_j \qquad\qquad (3 \le j \le n). \end{aligned}$$

Consequently, we have

$$\langle u_1^4 \rangle = \langle (u_1')^4 \rangle = \frac{1}{4} \langle (u_1 + u_2)^4 \rangle = \frac{1}{2} \Big(\langle u_1^4 \rangle + 3 \langle u_1^2 u_2^2 \rangle \Big).$$

This yields the required second relation:

$$\langle u_i^4 \rangle = 3 \langle u_i^2 u_j^2 \rangle. \qquad (A.4)$$

Solving (A.3) and (A.4), we get

$$\langle u_i^4 \rangle = \frac{3}{n(n+2)}. \qquad (A.5)$$

The same sort of technique could be used to compute the higher moments. However, in Weyl's paper there is a cleverer technique (probably known much earlier), which is a generalization of the method of Section A.1 that was used to compute the volume of $S^{n-1}(1)$. The idea is to compute

$$I = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} u_{j_1}^{i_1} \cdots u_{j_s}^{i_s} \exp\left(- u_1^2 - \cdots - u_n^2 \right) du_1 \ldots du_n$$

in two different ways. Define

$$q) = 1 \cdot 3 \cdots (q-1)$$

for even $q \geq 2$, and put $0) = 1$.

Theorem 1.5. *Let $i_1, \ldots, i_n$ be even integers and write $c = i_1 + \cdots + i_n$. Then*

$$\left\langle u_{j_1}^{i_1} \cdots u_{j_n}^{i_n} \right\rangle = \frac{i_1) \cdots i_n)}{n(n+2) \cdots (n+c-2)}.$$

Proof. It follows from the definition of the gamma function that

$$I = \prod_{p=1}^{n} \int_{-\infty}^{\infty} e^{-t^2} t^{i_p} dt = \prod_{p=1}^{n} \Gamma\left(\frac{i_p+1}{2}\right). \tag{A.6}$$

On the other hand, we can use polar coordinates to compute I. Denote by a_i the restriction of u_i to $S^{n-1}(1)$; then $u_i = a_i r$. Changing from rectangular to polar coordinates and using (A.1) and (A.2), we find that

$$\begin{aligned} I &= \int_0^{\infty} \int_{S^{n-1}(1)} (a_{j_1} r)^{i_1} \cdots (a_{j_n} r)^{i_n} e^{-r^2} r^{n-1}\, dA\, dr \qquad \text{(A.7)} \\ &= \left\{ \int_{S^{n-1}(1)} a_{j_1}^{i_1} \cdots a_{j_n}^{i_n}\, dA \right\} \int_0^{\infty} e^{-r^2} r^{c+n-1} dr \\ &= \left\langle a_{j_1}^{i_1} \cdots a_{j_n}^{i_n} \right\rangle \operatorname{volume}(S^{n-1}(1)) \frac{1}{2} \Gamma\left(\frac{c+n}{2}\right) \\ &= \left\langle a_{j_1}^{i_1} \cdots a_{j_n}^{i_n} \right\rangle \frac{\Gamma\left(\dfrac{c+n}{2}\right) \pi^{n/2}}{\Gamma\left(\dfrac{n}{2}\right)}. \end{aligned}$$

Therefore, from (A.6) and (A.7) we have

$$\begin{aligned}\left\langle a_{j_1}^{i_1} \cdots a_{j_n}^{i_n} \right\rangle &= \left\langle u_{j_1}^{i_1} \cdots u_{j_n}^{i_n} \right\rangle \\ &= \frac{\Gamma\left(\frac{n}{2}\right) \prod_{p=1}^{n} \Gamma\left(\frac{i_p+1}{2}\right)}{\Gamma\left(\frac{n+c}{2}\right) \pi^{n/2}} \\ &= \frac{\left(\frac{n}{2}-1\right)! \prod_{p=1}^{n} \left(\frac{i_p-1}{2}\right)!}{\left(\frac{n+c}{2}-1\right)! \pi^{n/2}}.\end{aligned} \tag{A.8}$$

But

$$\left(\frac{i_p-1}{2}\right)! = \left(\frac{i_p-1}{2}\right)\left(\frac{i_p-3}{2}\right)\cdots\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)! = \frac{i_p)\sqrt{\pi}}{2^{i_p/2}} \tag{A.9}$$

and

$$\begin{aligned}\frac{\left(\frac{n+c}{2}-1\right)!}{\left(\frac{n}{2}-1\right)!} &= \left(\frac{n+c-2}{2}\right)\left(\frac{n+c-4}{2}\right)\cdots\frac{n}{2} \\ &= \frac{n(n+2)\cdots(n+c-2)}{2^{c/2}}.\end{aligned} \tag{A.10}$$

Then the theorem follows from (A.8)–(A.10). □

A.3 Explicit Computation of the Volume $V_m^{\mathbb{K}^n(\lambda)}(r)$ of a Geodesic Ball in a Space $\mathbb{K}^n(\lambda)$ of Constant Curvature

As shown in Corollary 3.18, the volume of a geodesic ball of radius r in a space $\mathbb{K}^n(\lambda)$ of constant curvature λ is given by

$$V_m^{\mathbb{K}^n(\lambda)}(r) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \int_0^r \left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-1} dt. \tag{A.11}$$

For general n it would be very cumbersome to carry out the integration of the right hand side of (A.11). However, for any specific value of n the integration is elementary, though tedious. The following table gives the exact expressions for $V_m^{\mathbb{K}^n(\lambda)}(r)$ for $1 \le n \le 6$.

n	$\mathrm{V}_m^{\mathbb{K}^n(\lambda)}(r) = \frac{2\pi^{n/2}}{\Gamma(n/2)}\int_0^r \left(\frac{\sin t\sqrt{\lambda}}{\sqrt{\lambda}}\right)^{n-1} dt$	Power series expansion
1	$2r$	$2r$
2	$\frac{2\pi}{\lambda}\left(1-\cos(r\sqrt{\lambda})\right)$	$\pi r^2 - \frac{\pi\lambda r^4}{12} + \frac{\pi\lambda^2 r^6}{360} + O(r^7)$
3	$\frac{\pi}{\lambda}\left(2r - \frac{\sin(2\sqrt{\lambda}r)}{\sqrt{\lambda}}\right)$	$\frac{4\pi r^3}{3} - \frac{4\pi\lambda r^5}{15} + \frac{8\pi\lambda^2 r^7}{315} + O(r^8)$
4	$\frac{\pi^2}{6\lambda^2}\left(8 - 9\cos(r\sqrt{\lambda}) + \cos(3r\sqrt{\lambda})\right)$	$\frac{\pi^2 r^4}{2} - \frac{\pi^2\lambda r^6}{6} + \frac{13\pi^2\lambda^2 r^8}{480} + O(r^9)$
5	$\frac{\pi^2}{12\lambda^2}\left(12r - \frac{8\sin(2r\sqrt{\lambda})}{\sqrt{\lambda}} + \frac{\sin(4r\sqrt{\lambda})}{\sqrt{\lambda}}\right)$	$\frac{8\pi^2 r^5}{15} - \frac{16\pi^2\lambda r^7}{63} + \frac{8\pi^2\lambda^2 r^9}{135} + O(r^{10})$
6	$\frac{\pi^3}{240\lambda^3}\Big(128 - 150\cos(r\sqrt{\lambda})$ $+ 25\cos(3r\sqrt{\lambda}) - 3\cos(5r\sqrt{\lambda})\Big)$	$\frac{\pi^3 r^6}{6} - \frac{5\pi^3\lambda r^8}{48} + \frac{23\pi^3\lambda^2 r^{10}}{720} + O(r^{11})$

The following *Mathematica* function can be used to compute $\mathrm{V}_m^{\mathbb{K}^n(\lambda)}(r)$:

```
volsphere[n_,lambda_][r_]:=
Cancel[PowerExpand[2Pi^(n/2)/Gamma[n/2]Integrate[
(Sin[t Sqrt[lambda]]/Sqrt[lambda])^(n - 1),{t,0,r}]]]
```

Then the first column in the above table can be generated with

```
Table[volsphere[n,lambda][r],{n,1,6}]
```

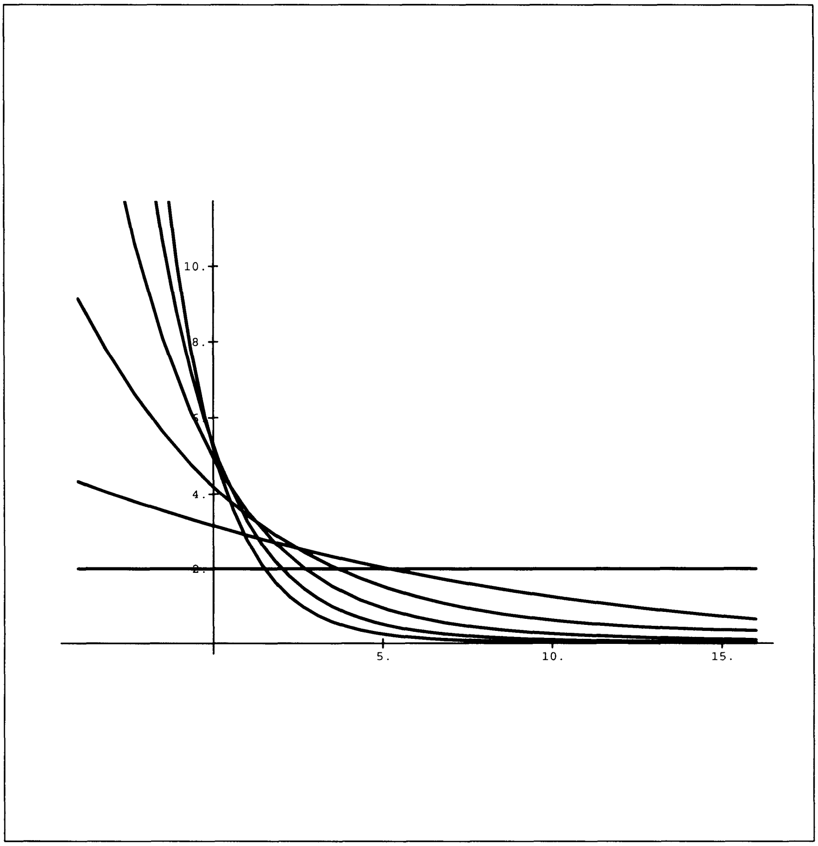
and the second column can be generated with

```
Table[Series[volsphere[n,lambda][r],
                    {r,0,n + 4}],{n,1,6}]
```

Also of interest is the graph of the function $\lambda \longrightarrow V_m^{\mathbb{K}^n(\lambda)}(r)$ for fixed r, say $r = 1$. We allow λ to be negative, pass through 0, and become positive. For $n \geq 2$ each of the functions $\lambda \longrightarrow V_m^{\mathbb{K}^n(\lambda)}(1)$ has infinitely many zeros. Here is the graph of

$$\lambda \longmapsto V_m^{\mathbb{K}^n(\lambda)}(1) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \int_0^1 \left(\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}\right)^{n-1} dt$$

for $-4 \leq \lambda \leq 16$ and $1 \leq n \leq 6$.



This plot can be generated in *Mathematica* with the command

```
Plot[Evaluate[Table[volsphere[n,lambda][1],{n,1,6}]],
{lambda,-4,16}]
```

Appendix B

In this section we describe some *Mathematica* routines for drawing tubes about curves in $\mathbb{R}^3$. For more information see [Gr17] and the site

http://math.cl.uh.edu/∼gray

maintained by M.J. Mezzino.

Mathematica versions of **T**, **N** and **B** are given by

```
unit[x_]:= x/Sqrt[x.x]
TT[alpha_][t_]:= unit[D[alpha[tt],tt]]/.tt->t
NN[alpha_][t_]:= unit[D[TT[alpha][tt],tt]]/.tt->t
BB[alpha_][t_]:= Cross[TT[alpha][t],NN[alpha][t]]
```

These definitions reflect the standard mathematical ones, and are adequate for the purposes illustrated below. The Simplify command should be inserted at strategic points if inspection of the various vector-valued functions is required.

Parametrization of a tube of radius **r** about a curve **gamma** in $\mathbb{R}^3$ can now be constructed as follows:

```
tubecurve[gamma_][r_][t_,th_]:= gamma[t] +
          r(-Cos[th]NN[gamma][t]+Sin[th]BB[gamma][t])
```

The code below was used to illustrate Problem 1.3 of Chapter 1, though the Show command is now set up to combine the the helix and its tube in the same picture:

```
n := 50
m:=18
helix[a_, b_, t_] := {a Cos[t], a Sin[t], b t}
gamma[t_] := helix[1, .3, t]
plot1 := ParametricPlot3D[Append[helix[1, .3, t],
       {Thickness[.01], Hue[0]}], { t, -.3, 5.3 Pi},
        PlotPoints -> n, Compiled -> False,
        DisplayFunction -> Identity]
plot2 := ParametricPlot3D[tubecurve[gamma][.4][t, th],
       {t, 0, 5  Pi}, {th, 0, 2 Pi},
    PlotPoints -> {n,m}, Compiled -> False,
       DisplayFunction -> Identity]
Show[plot1, plot2, Axes -> False,
       DisplayFunction -> $DisplayFunction]
```

A smaller value of n will allow plotting to take place more quickly, and is recommended for 'draft' plots.

Here is the program used to draw the frontspiece picture of 'conical tubes' on a twisted cubic:

```
n:= 30
gamma[t_]:= {t,t^2,t^3}
plot1:= ParametricPlot3D[Append[gamma[t],
        {Thickness[.01t^2],Hue[.9]}], {t,-1,1},
        Compiled->False,
        DisplayFunction->Identity]
plot2:= ParametricPlot3D[tubecurve[gamma][.15t^2][t,th],
        {t,-.9,.9}, {th,Pi/2,2Pi},
        Compiled->False, PlotPoints->n,
        DisplayFunction->Identity]
plot3:= ParametricPlot3D[tubecurve[gamma][.4t^2][t,th],
        {t,-.8,.8}, {th,Pi/2,2Pi},
        Compiled->False, PlotPoints->n,
        DisplayFunction->Identity]
Show[plot1, plot2, plot3, Boxed->False, Axes->False,
     ViewPoint->{4, 0, 1},
     DisplayFunction->$DisplayFunction]
```

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`http://math.cl.uh.edu/~gray`

maintained by M.J. Mezzino in memory of Alfred Gray.

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